

Revenue Management with End-of-Period Discounts in the Presence of Customer Learning

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Abstract

Consider a firm that sells identical products over a series of selling periods (e.g., weekly all-inclusive vacations at the same resort). To stimulate demand and enhance revenue, in some periods, the firm may choose to offer a part of its available inventory at a discount. As customers learn to expect such discounts, a fraction may wait rather than purchase at a regular price. A problem the firm faces is how to incorporate this waiting and learning into its revenue management decisions.

To address this problem we summarize two types of learning behaviors and propose a general model that allows for both stochastic consumer demand and stochastic waiting. For the case with two customer classes we develop a novel solution approach to the resulting dynamic program. We then examine two simplified models, where either the demand or the waiting behavior are deterministic, and present the solution in a closed form. We extend the model to incorporate three customer classes and discuss the effects of overselling the capacity and bumping customers. Through numerical simulations we study the value of offering end-of-period deals optimally and analyze how this value changes under different consumer behavior and demand scenarios.

1 Introduction

The rapid growth of online purchases of airline tickets and other travel-related products has presented the travel and leisure industry with a number of challenges and opportunities. These include the need to develop the capability to rapidly change prices and availability of inventory, track and respond to competitor moves, and address changes in consumer behavior. This growth has also provided the capability to offer inventory that is not selling at the expected rates, so called “distressed” inventory, at a discount in the days prior to a departure of a flight or other product: hotel room night, vacation package, weekend car rental, etc. Although frequently referred to as the “last minute” deals, such discounts are often offered days or weeks prior to the departure; for example, the lowest airfares are observed 3-8 weeks prior to departure, Stringer (2002).

Such end-of-period discounts present an opportunity to purchase products at noticeably lower prices and consumers are taking advantage of the offers. For example, a quick Web search for a week-long all-inclusive vacation for an approaching weekend in the Fall of 2009 produced choices for less than \$400; this compares with prices in excess of \$1,000 offered for the vacation months in advance. Increasingly, many travelers are learning to expect such end-of-period discounts and “prefer to book later in the hope of getting a good deal” (Fenton and Griffin 2004). According to American Express, “nearly half of all travelers say they intend to wait until the last minute to plan their vacations” (De Lisser 2002). Similarly, in private conversations, executives of a leading vacation tour operator noted that as a result of customer waiting for deep discounts, early bookings are “slow” and 27% of the bookings are made in the last 15 days. That is, travel firms are observing that as customers increasingly expect end-of-period discounts, additional inventory is distressed. As a result firms sell more units at a discount and lose revenue. This suggests that firms should carefully consider consumer response and incorporate it into their management policies.

The goal of this paper is to develop a stylized model that incorporates consumer response to revenue management. In this paper, we assume a travel firm (airline, car rental firm, hotel, etc.) needs to determine the number of units (seats, cars, rooms, etc.) to place on sale at the end of each of number of a selling seasons (flights, weeks, etc.) that we refer to as periods. (Note, that the term period refers to different flights, not the slices of time during the sales of a single flight.) The objective of the firm is to maximize the total discounted revenue over the horizon. In each period, a fraction of the customers purchase at a regular,

non-discounted price and a fraction waits for a potential end-of-period sale. Customers who wait, but do not receive inventory at the discounted price, may be offered inventory for purchase at a higher price at the end of the same period. The decision made by the firm is to determine the number of units (if any) to put on sale in each period, understanding that the fraction of customers that wait for a sale in the following period is affected by the decision in the current period.

We assume the discounted price is fixed, and the firm determines the number of units available at this price. Such a quantity-oriented approach is dominant in the travel industry as opposed to a price-changing approach which is more common in other industries such as retailing (Talluri and van Ryzin 2004). We consider model variations with two and three prices in order to derive insights about the optimal policies. In practice, one would implement revenue management with more price levels, e.g., hotels and cruise lines typically use 6-8*. Our initial model with two prices (and two customer classes) models such industries as packaged tours and performance events, where a common practice of pre-publishing prices in catalogs effectively reduces the firms' ability to increase prices in the case of high demand. Our subsequent model with three classes captures the examples of airlines, car rental firms, and hotels that can increase their price if there are customers willing to spend more for the product.

The fundamental contribution of this paper to the literature is the proposal and solution of a model of revenue management that incorporates consumer response to a firm's policy. Our model is distinguished from previous work in a number of directions. First and foremost, we model a series of repetitive revenue management decisions that influence customers' behavior for the future periods as consumers learn in this multiple period environment. The vast majority of the literature considers a single selling period (flight, etc.), and effectively ignores the possible effects on future periods. Second, we incorporate the double uncertainty of stochastic demand and stochastic consumer behavior. We show that the resulting dynamic programming model is not amenable to standard solution methodologies and make a theoretical contribution by presenting a solution methodology to a subclass of dynamic programs in which the state of the system evolves non-monotonically. Third, we derive the optimal, closed form policies for several simplified models, and through numerical studies we document the degree to which the optimal solution provides benefits

*Source: private conversation with a manager at a leading revenue management software firm.

over reasonable rule-based heuristics. Finally, we discuss the effects of different patterns of behavior with respect to the types and speed of learning, and with respect to the allowance of customer bumping.

The paper generates several managerial insights. First, we show that in the long-run a firm benefits by offering end-of-period discounts, even when consumers learn and react to the firm's decisions by waiting. The optimal policy, however, greatly depends on the way customers learn. If customers self-regulate their waiting behavior (e.g., recognizing that if all customers wait, no discounts will be provided), then the firm's policy is a passive one, where the firm puts some inventory on sale in every period. However, if the fraction of customers waiting for a discount evolves by interpolating or smoothing between its current state and some measure of the number of units placed on sale, then the firm takes an active policy. We show that a "bang-bang" policy, where the firm intermixes periods with many units on sales with those with none, is optimal. Second, we show that allowing some inventory to perish (and not be sold) may be more profitable than selling it at a discount. Third, we show that even in the absence of no-shows, overselling the capacity and bumping passengers may be an important factor in managing consumer behavior. Finally, we demonstrate that an intuitive heuristic that makes inventory available at a discount when regular sales are low performs rather poorly; in fact it does the opposite of what should be done when consumers learn and react to firms' inventory policies.

The remainder of the paper is organized as follows. In Section 2 we position our model in the body of relevant literature. In Section 3 we introduce the model and discuss key assumptions. In Section 4 we study the optimal policy for two customer classes under the assumption of "self-regulating" learning, and present the solution to the resulting dynamic program. The case of "smoothing" learning is discussed in Section 5, where we introduce two simplifications to our general model and present their optimal policies in the closed form. In Section 6 we extend our models to the case with three customer classes, and discuss the effects of bumping on customer behavior and on the resulting optimal policy of the firm. Numerical results are presented in Section 7, followed by the conclusions and prospects for future research. The paper is accompanied by online Appendices that contain additional model discussions, proofs, and two model extensions.

2 Literature Review

As revenue management has been an active area of research, we review only the literature directly related to the current study. McGill and van Ryzin (1999), Bitran and Caldentey (2003) and the book by Talluri and van Ryzin (2004) provide comprehensive reviews of the broader literature. Most previous research focusses on pricing and inventory policy for either a single flight or product, or a network of flights or multiple products, where strategic response by customers to the determined policy is ignored.

Relevant work that considers multiple selling seasons includes papers on intertemporal price discrimination and advance selling, such as Stokey (1979), Sobel (1984), and Conlisk et al. (1984). A summary of retail pricing can be found in Lazear (1986). Besanko and Winston (1990) and Gale and Holmes (1993) discuss the optimal price skimming by a monopolist. Dana (1999), Xie and Shugan (2001), and Tang et al. (2004) discuss advance selling. These works do not consider customer learning and in this regard are different from ours. Customer learning is often modeled through reference price effects; for example, consider Greenleaf (1995) and Popescu and Wu (2007). These models do not consider the internal dynamics of selling to several classes of customers within each selling season, which is a major feature of revenue management systems. Sen and Zhang (1999) consider a newsvendor problem with two customer classes whose price assumptions are similar to our single period model; they do not study repetitive problems and customer learning.

Recently, a number of papers (Aviv and Pazgal 2008, Cachon and Swinney 2009, Elmaghraby et al. 2008, Gallego et al. 2008, Liu and van Ryzin 2008, Zhang and Cooper 2008) consider customer behavior with respect to markdown policies. In our model the firm does not pre-commit to a price path as in these papers. We allow the firm to decide whether there should be a markdown, markup or both: a markdown could happen if all unsold inventory is put on sale, a markup could happen if no inventory is put on sale, or the price path could be a markdown followed by a markup. Further, these papers consider only a single selling period.

Several papers allow for both markups and markdowns. Asvanunt and Kachani (2007) consider the purchasing strategy of a single strategic consumer and model her waiting decision as an optimal stopping problem. They also report an empirical study demonstrating practical effectiveness of their solution. Their extension to the case with multiple strategic consumers assumes that consumers are oblivious to other strategic buyers. Levin et

al. (2010) allow each consumer to consider their effect on the others, but assume that the future utility of a unit is the same for all consumers. In contrast we assume that customers' willingness-to-pay is constant through the selling season and that the firm rations the discounted units.

Su (2007) considers a game between the firm and a fraction of consumers who act strategically and shows how the size of this fraction determines whether the firm should increase or decrease the price. Anderson and Wilson (2006) also assume that an exogenous fraction of customers will wait. These works, however, do not specify how the fraction of customers who act strategically is determined. In contrast, this is a key feature of our model, where a fraction of customers who wait is changing, as consumers learn from past decisions of the firm.

All of the above mentioned works assume a single period in which the derived equilibrium policies of the firms and customers are arrived at after a process of learning and reacting. These works do not recognize the possible dependency between the outcome of one selling season and the behavior of consumers in the future seasons, conveyed, for example, via news articles, information on websites or word-of-mouth. Our work presents a model with multiple selling periods and captures this dependency.

Three works consider a multiple-period setting similar to ours. Cooper et al. (2006) model the "spiral-down" effect and demonstrate that in a multiple-period problem with customer learning the effect of otherwise optimal (single-period) revenue management policy could be significantly diluted. Gallego et al. (2008) investigate a model in which a firm's decision about the number of units sold at a discount alters consumers' expectations about the probability of acquiring a discounted item in the future. They demonstrate numerically that the optimal inventory policies could evolve from one period to the next in a complex fashion that cannot be captured by a single period model. Liu and van Ryzin (2009) consider a case where consumers learn about firm's capacity using a moving average smoothing process, and decide whether to wait for a possible markdown or not. In contrast with the previous paper, however, they find that the optimal policy converges to either a rationing equilibrium (with a high early purchase price and a low markdown price) or a low-price-only equilibrium. Our approach is more general than the above: we do not limit ourselves to markdowns, consider alternative learning processes, incorporate the effects of overselling/bumping on consumers' propensity to wait, and allow for stochastic demand

from multiple customer classes. We present our model next.

3 Model

Consider a sequence of identical offerings of a perishable product or service, for example, weekly all-inclusive vacations at the same resort, Wednesday morning flights from London to New York, or weekend car rentals. To distinguish between the copies of the product offerings, we say that they are offered in different *periods*, and there is one offering per period. In each period $t = 1, 2, \dots, T$, for finite T , there are N units of product available. For simplicity, we treat all demands and capacities as continuous variables, and assume that for $\Delta > 0$, Δ units of inventory fill Δ units of demand.

We initially assume that there are two prices, $p_2 \geq p_1$, and two customer classes, such that p_i , $i = 1, 2$, is the highest price that class i is willing to pay for the unit of product, i.e., class-1 is the lower class customer and class-2 is the higher class. Later, in Section 6, we present an extension to three prices/customer classes.

Let $Q_t^i(Y_t)$, $i = 1, 2$ be the demand for class- i and let $D_t(Y_t) = Q_t^1(Y_t) + Q_t^2(Y_t)$ be the total demand (at price p_1 , note class-1 customers do not purchase at p_2). Here Y_t is a random demand parameter, reflecting, for example, weather, exchange rate or other exogenous factors that influence the demand. Since naturally demand from different classes can depend differently on those factors, we assume that $Y_t = \{Y_t^j\}_{j=1}^J$ is a vector of random variables with joint CDF $F_Y(y)$, defined on a finite, convex set $\mathcal{Y} \in \mathfrak{R}^J$. Such construct allows us to capture correlations between demand classes; see Section 8. We assume $Q_t^2(Y_t) < N$ for all $Y_t \in \mathcal{Y}$ in order to avoid trivial cases when all capacity is sold and hence no discounts and learning are possible.

As in other models in the literature, some class-2 customers may wait for a discount that may be offered to attract class-1 customers at the end of the period. In line with earlier works (e.g., Pfeifer 1989) we refer to these diverting customers as “shoppers.” However, a key feature of our model is that their waiting behavior changes over time in response to the firm’s decisions. We assume that consumer waiting behavior is described by a *waiting parameter*, θ_t , that represents the propensity of customers to wait for a discount. For example, there could be a random fraction, α_t , of class-2 customers waiting and θ_t could be the average fraction waiting (this is the construct we use in our simplified models). The

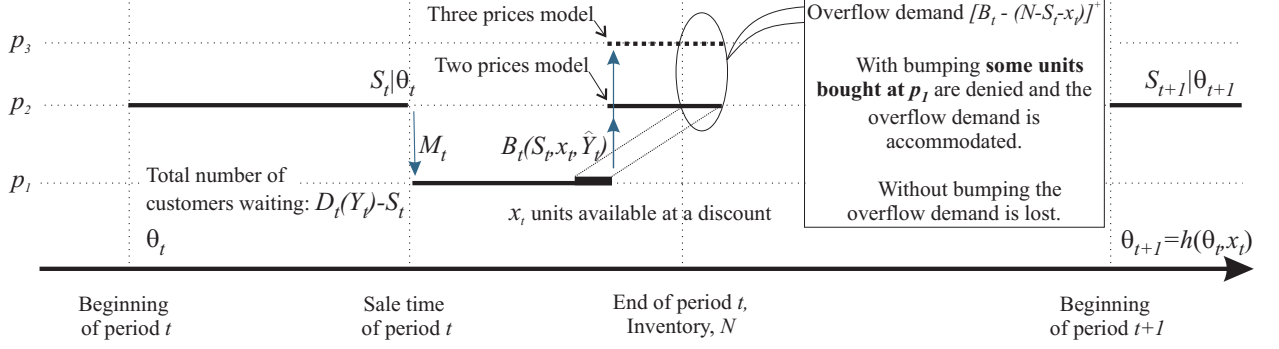


Figure 1: Timeline of the model.

waiting parameter θ_t changes over time capturing customer learning. We assume that θ_t is known at the beginning of period t . The dynamics of the selling period are depicted in Figure 1.

The firm initially sets the price at p_2 and observes the initial sales, S_t to class-2 customers. The remainder of class-2 customers, M_t , are shoppers who wait for a potential discount towards the end of the period; note $S_t + M_t = Q_t^2$. Class-1 customers as per Cachon and Swinney (2009) are “bargain-hunters”: they do not act strategically and rather all wait and purchase only if price p_1 is offered. The total demand at price p_1 therefore is $D_t(Y_t) - S_t \equiv Q_t^1 + M_t$.

At a point in time close to the end of period t , denoted as the sale time, the firm determines x_t , the number of unsold units to put on sale at price p_1 , $x_t \geq 0$. If $x_t \geq Q_t^1 + M_t$ then all waiting demand is satisfied and the firm receives revenue of $p_2 S_t + p_1 x_t$ and moves to the next period. Otherwise, the firm can receive additional revenue by accommodating the unserved class-2 shoppers as follows.

Let $B_t(S_t, x_t, Y_t)$ be the number of unserved class-2 customers. We assume that once all x_t discounted units are sold, the firm again offers units for sale at price p_2 to accommodate unserved class-2 shoppers (or in the three prices extension, Section 6, at a higher price p_3). In some industries, for example in airlines, it is common to oversell the capacity, forcing some customers to be “bumped”, i.e., displaced from their seat by another, typically higher paying, customer. In such cases *with bumping* we assume that if B_t exceeds the number of unsold units, $(N - S_t - x_t)$, customers that purchased at a discount are bumped; their units are sold at price p_2 (or p_3 in the three prices model) and the firm incurs a penalty p_C per unit bumped. In other industries, for example in cruise lines, only unsold units may be sold

at this time. This may occur because of competitive norms, legislation or regulation, or for logistical reasons, such as when a physical object is sold. Letting $p_C = p_2$ (or p_3 in the three prices model) accomplishes this goal as there is no economic justification for overselling. Thus in cases *without bumping* the potential demand, $B_t - (N - S_t - x_t)$ is lost.

Observe that there are incentives for class-2 customers to purchase early. Without bumping class-2 customers may not be served and presumably they would prefer to purchase at p_2 rather than not. With bumping, class-2 customers that purchase a discount seat are subject to bumping. We assume that bumped customers cannot repurchase a unit in the same period – this is a reasonable assumption supported by discussions with travel industry personnel (see model discussion in Appendix A).

The total revenue of the firm net the bumping cost for period t given (S_t, x_t, Y_t) is

$$g_t(S_t, x_t, Y_t) = p_2 S_t + p_1 \min[x_t, D_t(Y_t) - S_t] + p_2 B_t(S_t, x_t, Y_t) - p_C (B_t(S_t, x_t, Y_t) - (N - S_t - x_t))^+ . \quad (1)$$

For notational convenience let $\hat{Y}_t \equiv Y_t | (\theta_t, S_t)$ represent the conditional random vector Y_t having observed S_t and knowing θ_t . The expected single-period revenue in period t is then given by:

$$r_t(\theta_t, S_t, x_t) = \int_{\mathcal{Y}} g_t(S_t, x_t, y) dF_{\hat{Y}_t}(y). \quad (2)$$

where $F_{\hat{Y}_t}(y) \equiv F_{Y_t | (\theta_t, S_t)}(y)$ is the cdf of \hat{Y}_t .

We refer to the 2-vector (θ_t, S_t) as to the *state of the system*. We assume that the system evolves based on the decision x_t according to a function $h(\theta_t, x_t)$, defining θ_{t+1} , and a random draw of S_{t+1} from the distribution of future sales, $F_{S_{t+1} | \theta_{t+1}}(\cdot)$. We refer to $h(\cdot)$ as the *learning function* because it reflects the changes in the waiting behavior of the customers as they learn about the policy of the firm. We define two types of the learning functions:

- (i) *smoothing*, if $\frac{\partial h}{\partial x} \geq 0$ and $\frac{\partial h}{\partial \theta} \geq 0$, e.g., $h(\theta, x) = \lambda \frac{x}{N} + (1 - \lambda)\theta$ for $0 \leq \lambda \leq 1$;
- (ii) *self-regulating*, if $\frac{\partial h}{\partial x} \geq 0$ and $\frac{\partial h}{\partial \theta} \leq 0$, e.g., $h(\theta, x) = \kappa + \lambda \frac{x}{N} - (1 - \lambda)\theta$ for $\{\kappa, \lambda > 0 | \kappa + \lambda \leq 1\}$.

We note that the current literature, e.g., Popescu and Wu 2007, Liu and van Ryzin 2009, considers smoothing dynamics in the parameters of the aggregate demand model. By presenting alternative learning mechanisms, we consider broader cases and derive additional insight; see more discussion about learning behaviors in Section 8.

The objective of the firm is to maximize the expected T -period revenue, discounted at a fixed rate $\delta \in (0, 1)$. Therefore, the firm determines the number of units on sale, x_t , for each period $t = 1, 2, \dots, T$ and some initial θ_1 , by solving the following dynamic program

$$J_t(\theta_t, S_t, x_t) = r_t(\theta_t, S_t, x_t) + \delta E_{S_{t+1}|\theta_{t+1}} [J_{t+1}^*(\theta_{t+1}, S_{t+1})] \quad (3a)$$

subject to

$$\begin{aligned} J_t^*(\theta_t, S_t) &= \max_{0 \leq x_t \leq N - S_t} J_t(\theta_t, S_t, x_t), \\ \theta_{t+1} &= h_t(\theta_t, x_t), \\ J_{T+1}^*(\theta_{T+1}, S_{T+1}) &= 0 \text{ for all } (\theta_{T+1}, S_{T+1}). \end{aligned} \quad (3b)$$

We refer to (3) as to the *general* model, as it expresses the uncertainty in the overall demand as well as in the fraction of class-2 customers waiting in every period. In Sections 5.1 and 6 we consider *simplified* models, where one of these uncertainties is removed.

We approach the problem by establishing properties of $J_t(\theta_t, S_t, x_t)$, in particular concavity and supermodularity. To do so, we make several assumptions on the demand and stochastic ordering of the random variables. Specifically, recall that a random variable X with cdf $F_X(x, \theta)$ is said to be stochastically increasing (decreasing, convex, concave, supermodular, etc.) in parameter θ iff $1 - F_X(x, \theta)$ is increasing (decreasing, convex, concave, supermodular, etc.) See sections 2.2 and 3.9.1 in Topkis (1998) as well as Shaked and Shantikumar (1994) for more discussion about stochastic ordering, examples, and applications. We assume that demand function $D_t(Y_t)$ is increasing in Y_t . We assume that (i) S_t is stochastically decreasing in θ_t ; (ii) \hat{Y}_t is stochastically increasing in S_t ; and (iii) \hat{Y}_t is stochastically increasing in θ_t .

These assumptions are consistent with the following intuitive observations. Since θ_t measures the propensity of customers to wait, the number of customers who purchase (i.e., do not wait), S_t , should decrease in θ_t . Similarly, more customers purchasing at the initial price implies increased $Q_t^2(Y_t)$. Defining Q_t^2 to be increasing in at least one element of Y_t and not decreasing in any, implies \hat{Y}_t increasing in S_t . The distribution of the conditional random variable \hat{Y} is dependent on θ . Upon observation of the same sales, S_t , and increase in θ would imply greater class-2 demand leading to a (stochastic) increase in \hat{Y} .

For more discussion on the various elements and assumptions of our model (aggregate demand, waiting parameter, bumping, allocation of discounted unit, discount price, and learning behaviors) please refer to Appendix A. In what follows we analyze the model; first for the case of self-regulating learning, and then for smoothing.

For the case of self-regulating learning we show that the standard methodologies for establishing concavity, such as Topkis (1998) Section 3.9.2, Putterman (1994) Section 4.7.3, or their recent extensions, e.g., Smith and McCardle (2002), are not applicable. In their models the state of the system evolves monotonically in the previous state and decision. That is, the components of the state vector either increase or decrease in the previous state and decision. In our general model (3), state transitions are not monotonic, since S_t is stochastically decreasing in θ_t , while θ_t is increasing in either θ_{t-1} or x_{t-1} , or both. We therefore develop the necessary methodology and show concavity in Section 4.

In Section 5 we study the case with a smoothing learning function and show that in general concavity does not hold, unless the speed of customer learning is “slow.” Then we consider two simplifications to the general model and derive their solutions in closed form. Section 6 presents a model with three customer classes.

4 Optimal Policy for Self-Regulating Learning

In this section we derive the conditions under which the revenue-to-go function is concave when the learning function, $h(\cdot)$, is self-regulating. Concavity implies that the optimal policy for the firm places some units on sale in each period, relying on the consumer behavior regulating the number of customers waiting.

We organize this section as follows. First in Section 4.1 we show that $J_t(\theta_t, S_t, x_t)$ is concave, supermodular and increasing under that assumption that these properties hold for the single-period expected revenue function, $r(\theta, S, x)$. Then in Section 4.2 we discuss the properties of the revenue function $g(S, x, \hat{y})$, given in (1) that ensure concavity, supermodularity and monotonicity of $r(\theta, S, x)$. We conclude by presenting an example.

4.1 Concavity in Dynamic Programs With Nonmonotonic State Transitions

Observe from (3a) that since $\theta_{t+1} = h_t(\theta_t, x_t)$ is independent of S_t , the expected future revenue, $E_{S_{t+1}|\theta_{t+1}} [J_{t+1}^*(\theta_{t+1}, S_{t+1})]$, does not depend on S_t and therefore, letting $\phi_{t+1}(h_t(\theta_t, x_t)) = E_{S_{t+1}|\theta_{t+1}} [J_{t+1}^*(\theta_{t+1}, S_{t+1})]$ we can substitute

$$J_t(\theta_t, S_t, x_t) = r_t(\theta_t, S_t, x_t) + \delta \phi_{t+1}(h_t(\theta_t, x_t)) \quad (4)$$

where the function ϕ_{t+1} can be interpreted as the expected future revenue.

We make the following four assumptions which we show hold in the next section. We assume that $r_t(\theta_t, S_t, x_t)$ is **(A1)** jointly concave in (θ_t, x_t) , **(A2)** supermodular in (θ_t, S_t, x_t) , **(A3)** increasing in S_t , and **(A4)** S_t is stochastically concave in θ_t (and recall that by assumption (i) from Section 3 it is also stochastically decreasing in θ_t). We also make the fundamental assumption of Section 4 **(A5)** $h_t(\theta_t, x_t)$ is linear self-regulating, i.e., $\frac{\partial h}{\partial x} \frac{\partial h}{\partial \theta} \leq 0$, $\frac{\partial^2 h}{\partial x^2} = \frac{\partial^2 h}{\partial \theta^2} = \frac{\partial^2 h}{\partial x \partial \theta} = 0$.

Concavity and supermodularity in (4) are related by the following Lemma (all proofs are presented in Appendix B):

Lemma 1 *If ϕ is concave in h , then J is concave in x and supermodular in (θ, S, x) .*

Therefore it is sufficient to show that ϕ is concave in h , which in our original notation corresponds to $E_{S_t|\theta_t} [J_t^*(\theta_t, S_t)]$ being concave in θ_t . Let $f_{S_t}(y; \theta_t)$ be the density of S_t given θ_t . Then $E_{S_t|\theta_t} [J_t^*(\theta_t, S_t)] = \int J_t^*(\theta_t, y) f_{S_t}(y; \theta_t) dy$. Concavity of this integral is established by the following Lemma, which extends Theorem 3.9.1 of Topkis (Theorem 7 in the Appendix) to the case where the integrand depends on the parameter.

Lemma 2 *Let a family of univariate random variables X_θ be stochastically decreasing and concave in scalar parameter θ , with cdf $F_X(x; \theta)$ twice differentiable in θ . Let $v(\theta, x)$ be supermodular in (θ, x) , increasing in x and twice differentiable and concave in θ . Then $\int v(\theta, x) dF_X(x; \theta)$ is concave in θ .*

As a simple example of a distribution family that satisfies the conditions of the above Lemma consider Uniform $[0, a - b\theta]$ for $a, b \geq 0, a - b\theta > 0$. Please refer to Shaked and Shantikumar (1994) for further discussion of how to construct random variables with the necessary properties.

Our main result in this section is given by the following Theorem.

Theorem 1 *Under assumptions **A1** – **A5**, $J_t(\theta_t, S_t, x_t)$ is concave in x_t and supermodular in (θ_t, S_t, x_t) for all $t = 1, 2, \dots, T$.*

Problem (3) defines a subclass of dynamic programs for which a vector of the system state is not monotonic in the previous period's state and decision. In this section we have shown how to establish concavity for such dynamic programs. A similar logic with a different set of initial assumptions could be used to prove other properties, for example, convexity.

We also note that to our knowledge there is no published research dealing with concavity (convexity) in the dynamic programs with nonmonotonic transitions. As such this is a technical contribution of our paper.

Next we discuss the underlying conditions on the customer behavior which ensure concavity.

4.2 Sufficient Conditions for Concavity of $J_t(\theta_t, S_t, x_t)$

In this section we study the properties of $B(S, x, \hat{Y})$ and the other parameters of the model that ensure that the expected single-period revenue function $r(\theta, S, x)$ is concave, supermodular, and increasing, so that by Theorem 1 the revenue-to-go, $J_t(\theta_t, S_t, x_t)$, is concave for every period. Since the discussion relates to a single period, time indices are omitted.

$B(S, x, \hat{Y})$ determines the number of class-2 “shoppers” who remain waiting once all discounted units are sold. That is, B is the initial number of class-2 shoppers, $Q^2(\hat{y}) - S$, minus those who bought at a discount, where the latter reflects allocation of discounted units. Therefore B should satisfy the following intuitive conditions: **B1**: B is increasing in \hat{Y} and decreasing in S and x ; **B2**: $\partial B / \partial x \geq -1$; **B3**: $\partial B / \partial S \geq -1$; **B4**: if $x \geq D(y) - S$ then $B(S, x, y) = 0$; and **B5**: $B(S, x, y)$ is piecewise concave in x on $[0, D(y) - S)$ and $[D(y) - S, N]$.

The increasing/decreasing properties of (B1) follow directly from the definition. Conditions (B2) and (B3) hold because the discounted units are allocated between class-1 and -2 customers: if additional Δ discounted units are available, some of them will be sold to class-1 customers, and so the number of class-2 customers remaining can decrease by at most Δ , and similarly, if additional Δ class-2 customers purchased early, then the chances of waiting class-2 customers to obtain discounted units decrease (because the number of class-1 customers does not change), thus the number of unserved class-2 customers decreases by at most Δ . (B4) implies if sufficient capacity is available to accommodate all customers, then no customers remain. Finally, (B5) implies the rate of decrease in the number of unsatisfied customers (weakly) increases with the number of discounted units, e.g., as would be the case if units are allocated randomly. In addition to assumption (ii) that \hat{Y} is stochastically increasing, we also assume that \hat{Y} is stochastically concave in S and θ , and stochastically supermodular in (θ, S) . These additional assumptions hold until the end of this section. We have the following Lemmas:

Lemma 3 $r(\theta, S, x)$ is increasing in S and concave in θ .

Lemma 4 $r(\theta, S, x)$ is concave in x if $D(\hat{y}) > N$ for all \hat{Y} or if $\frac{\partial B}{\partial x} \geq -p_1/p_2$.

Lemma 4 implies that if there is always sufficient demand to fill the capacity the revenue function is concave. Otherwise, the assumption $\partial B/\partial x \geq -p_1/p_2$ implies that for a Δ increase in x , at most $p_1/p_2 \times 100\%$ is allocated to class-2 customers and $(1 - p_1/p_2) \times 100\%$ goes to class-1. For example, if discounted inventory is allocated in proportion to demand, the ratio of class-1 to class-2 customers should be greater than $(p_2 - p_1)/p_2$. If demand is low and $\partial B/\partial x < -p_1/p_2$, then for each additional unit of x , the additional sales at the lower price cannibalize too much of the revenue from the waiting class-2 customers. In this case, offering no discounts is optimal, though concavity in the expected revenue cannot be assured. We note that concavity of $r(\theta, S, x)$ in x could hold in this case if the distribution of \hat{Y} places sufficiently small probability on $\mathbf{Y}' = [\hat{y} | D(\hat{y}) \leq N, \frac{\partial B}{\partial x} \leq -p_1/p_2]$.

The optimal number of units on sale in the single period can be determined by solving the first-order condition $\partial r(\theta, S, x)/\partial x = 0$, and is given by the following corollary:

Corollary 1 *The single-period optimal number of units on the end-of-period sale, x^* , satisfies*

$$\begin{aligned} p_1 \text{Prob}(D(\hat{Y}) > S + x^*) + p_2 \int_{y \setminus y_L} \frac{\partial B(S, x, y)}{\partial x} dF_{\hat{Y}}(y) \\ = p_C \left(\text{Prob}(B(S, x^*, \hat{Y}) > N - S - x^*) + \int_{y_H} \frac{\partial B(S, x, y)}{\partial x} dF_{\hat{Y}}(y) \right) \end{aligned} \quad (5)$$

where $y_L(S, x) = \{y : D(y) \leq S + x\}$ and $y_H(S, x) = \{y : B(S, x, y) > N - S - x\}$ are the sets of demand outcomes where all demand is either accommodated prior to any last minute price increase (y_L) or the unsatisfied class-2 demand exceeds the remaining capacity and thus some customers will be bumped (y_H).

Recalling $\partial B/\partial x < 0$, the first-order condition implies x should be chosen to balance the marginal additional revenue gained from bargain-hunters (class-1) less the marginal diverted revenue from class-2 customers who purchase at p_1 with the marginal penalty costs less the penalty costs saved by serving additional class-2 customers. That is, (5) expresses the newsvendor-like balance:

$$\begin{aligned} p_1 \text{Prob}((S + x^*)^{\text{th}} \text{ unit is sold at } p_1) - p_2 \text{Exp}[\text{Marginal decrease in } \# \text{ of diverted customers}] = \\ p_C (\text{Prob}(\text{Bumping a customer}) - \text{Exp}[\text{Marginal decrease in } \# \text{ of bumped customers}]). \end{aligned}$$

Joint concavity and supermodularity of $r(\theta, S, x)$ are discussed in the following Lemma:

Lemma 5 $r(\theta, S, x)$ is supermodular in (θ, S, x) and jointly concave in (θ, x) if either (i) $D(\hat{y}) > N$ for all \hat{y} in the support of \hat{Y} and $\frac{\partial B}{\partial x} = \text{const}$ ($\frac{\partial B}{\partial x} = -1$ if $p_C > 0$); or (ii) $D(\hat{y}) \leq N$ for all \hat{y} in the support of \hat{Y} , and $\frac{\partial B}{\partial x} = -p_1/p_2$.

Summarizing, we have the following theorem:

Theorem 2 *The revenue-to-go function is concave in x_t under conditions (B1) – (B5) if either (i) total demand always exceeds capacity and the marginal discounted unit is allocated between customers classes in a constant proportion (the marginal unit is sold to a class-2 customer if the penalty cost is positive); or (ii) total demand is always less than capacity and the marginal discounted unit is allocated between the classes in proportion to their revenues.*

As an example of a function that satisfies the assumptions, suppose that demands from class-1 and -2 customers are given by multiplying nominal demands, d_1 and d_2 by a random variable, Y . Demand at price p_2 is yd_2 and at price p_1 is $y(d_1 + d_2)$. Let \underline{y} be the minimum value of the support of Y . For the case with $p_C = 0$ and $\underline{y}(d_1 + d_2) \geq N$, consider $B(S, x, \hat{y}) \equiv \hat{y}d_2 - S - x\frac{d_2}{d_1+d_2}$. Here, the number of remaining class-2 customers reflects the total number of class-2 customers that wait, $M = \hat{y}d_2 - S$, net the number of class-2 customers that purchased product at the end-of-period sale. Further, the discounted units are allocated on proportion, which is constant and depends on the nominal demands, $d_2/(d_1 + d_2)$. With this, $g(S, x, \hat{y}) = \hat{y}p_2d_2 + x\left(p_1 - p_2\frac{d_2}{d_1+d_2}\right)$, which is supermodular in (S, x, \hat{y}) , and linear in (x, \hat{y}) . Therefore the conditions of Theorem 2 hold and the expected revenue-to-go function is concave for every period, and so the optimal number of units on sale is easy to find.

Our results also suggest a neat interpretation of the optimal policy for the case with self-regulating learning. First, concavity implies that the firm places some units on sale in every period. Furthermore, since the revenue function is supermodular, the number of units on sale increases in the waiting parameter. But, the self-regulating learning behavior controls the number of customers waiting in the subsequent period so that it does not continue to increase. The firm takes a passive role, placing some units on sale, and relying on the consumer behavior to control future waiting. This is not the case for smoothing learning functions, where the firm must actively manage consumer waiting as we discuss below.

5 Optimal Policy for Smoothing Learning Function

In this section we assume that the learning function $h_t(\theta_t, x_t)$ is smoothing; that is, the next period's waiting parameter, θ_{t+1} , increases in both the current waiting parameter, θ_t , and the number of units on sale in period t , x_t . We show that in the general model the revenue-to-go function is not necessarily concave, unless the speed of consumer learning is "slow" as defined below. To address the problems with arbitrary speed of learning we present two simplified models and show that for either simplification, the optimal policy has a "bang-bang" structure where the firm alternately places a number, \hat{x}_t , or zero units on sale. We describe this optimal policy in the closed form.

Under the assumption that the learning function is linear, concavity and supermodularity of the revenue-to-go require, respectively:

$$\frac{\partial^2}{\partial x^2} J(\theta, S, x) = \frac{\partial^2}{\partial x^2} r(\theta, S, x) + \delta \frac{\partial^2 \phi}{\partial h^2} \left(\frac{\partial h}{\partial x} \right)^2 \leq 0 \text{ and} \quad (6)$$

$$\frac{\partial^2}{\partial x \partial \theta} J(\theta, S, x) = \frac{\partial^2}{\partial x \partial \theta} r(\theta, S, x) + \delta \frac{\partial^2 \phi}{\partial h^2} \frac{\partial h}{\partial x} \frac{\partial h}{\partial \theta} \geq 0. \quad (7)$$

Because smoothing learning implies $\frac{\partial h}{\partial x} \frac{\partial h}{\partial \theta} \geq 0$, for (6) and (7) to hold when r is concave and supermodular, we must place some restrictions on the curvature of $\phi(h(\theta, x))$.[†] Specifically, observe that $\partial h / \partial x$ reflects the speed at which customers learn about the firm's decisions. If r is concave, then from (6) if $\partial h / \partial x$ is small enough then J would be also concave.

Slow learning has been documented in the works on reference price learning with respect to the sales promotions. Greenleaf (1995) and Hardie et al. (1993) studied point-of-sales data for such commodities as peanut butter and refrigerated orange juice, and reported an analog of our $\partial h / \partial x$ to be at 0.075 and 0.17 respectively. However, we know of no research regarding the speed of learning for discounts in services. This is of interest for future research.

The speed of learning also plays an important role in Gallego et al. (2008) and Liu and van Ryzin (2009). Specifically, in the former, numerical simulations suggest that in the case of slow learning, the firm should offer a constant number of units on sale, and otherwise the number should be raised and lowered in alternate periods. Next we prove a similar result analytically and provide a closed form expression for the number of units on sale.

[†]Remark: This was not necessary in the case with self-regulating learning because in that case $\frac{\partial h}{\partial x} \frac{\partial h}{\partial \theta} \leq 0$ and so when r and ϕ are concave, concavity and supermodularity of J also follow.

5.1 Simplified Models

In this section we simplify the general model so that upon observing the initial sales, the firm can infer the *exact* number of customers waiting for period t . The future demand and purchasing behavior remain stochastic. This simplification allows us to solve the problem in the closed form while utilizing a less constrained $B(S, x, y)$ and relaxing the linearity assumption of the learning function.

For the remainder of the paper, let $Y \in [\underline{y}, \bar{y}]$ be a random variable and let demand from each class reflect a nominal demand, d_i , $i = 1, 2$ multiplied by Y . For class-2, demand is $d_2 Y_t$ and for class-1, demand is $d_1(a + bY_t)$, where a, b are given constants. Observe that with $b > 0$ class-1 and -2 demands are positively correlated, with $b < 0$ then are negatively correlated and for $a > 0, b = 0$ class-1 demand is a constant. Note that in the two former cases demands are perfectly correlated. An extension to the non-perfectly correlated demands is presented in Appendix C. With these, the total demand from both classes is $D_t(Y_t) = Y_t d_2 + (a + bY_t)d_1$. We assume $\bar{y}d_2 \leq N$ and $d_2 + bd_1 \geq 0$ to ensure that D is increasing in Y .

Let $\alpha_t \in [\underline{\alpha}, \bar{\alpha}] \subseteq [0, 1]$ be the fraction of the class-2 demand that waits for the end-of-period sale in period t . We refer to α_t as the *waiting fraction*. Observe that $M_t = \alpha_t Y_t d_2$ and $S_t = (1 - \alpha_t) Y_t d_2$.

In this section we consider two simplifications:

- (i) **Deterministic waiting fraction model**, in which the demand multiplier, Y_t , is stochastic, but its distribution does not depend on the waiting parameter; the waiting fraction is deterministic with $\alpha_t \equiv \theta_t$, and evolves according to a smoothing concave learning function $\alpha_{t+1} = h_t(\alpha_t, x_t)$;
- (ii) **Deterministic demand model**, in which $Y_t \equiv \text{Constant}$ for all $t = 1, 2, \dots, T$ and w.l.o.g. we set $Y_t \equiv 1$; the random waiting fraction, α_t , is stochastically increasing and concave in the waiting parameter θ_t , which evolves according to a smoothing concave learning function $\theta_{t+1} = h_t(\theta_t, x_t)$.

Let $A_t(\alpha_t, Y_t)$ be the actual demand for the discounted seats. $A_t(\alpha_t, Y_t) = D_t(Y_t) - S_t = ad_1 + Y_t(d_2 + bd_1) - S_t$. Since the firm puts x_t units on the end-of-period sale, $\min[x_t, A_t]$ units are sold at the discounted price p_1 . Assuming that the discounted inventory is allocated proportionally between class-1 and class-2 customers based on their *realized* demands, the

number of class-2 customers that purchase discounted units is $\min[x_t, A_t] \frac{\alpha_t Y_t d_2}{A_t}$. The number of unserved class-2 customers after the sale is

$$B_t(\alpha_t, Y_t, x_t) = \alpha_t Y_t d_2 \left(1 - \frac{\min[x_t, A_t]}{A_t} \right). \quad (8)$$

and from (1) the net single-period revenue is

$$\begin{aligned} g_t(\alpha_t, Y_t, x_t) &= p_2 S_t + p_1 \min[x_t, A_t] + p_2 \alpha_t Y_t d_2 \left(1 - \frac{\min[x_t, A_t]}{A_t} \right) \\ &\quad - p_C \left(\alpha_t Y_t d_2 \left(1 - \frac{\min[x_t, A_t]}{A_t} \right) - (N - S_t - x_t) \right)^+ \end{aligned} \quad (9)$$

Observe that for either simplified model by knowing θ_t and observing S_t the firm can determine the exact (realized) values for α_t and y_t , and therefore at the sale time there is no uncertainty for the current period. If $A_t + S_t = D(y_t) < N$, then the firm has *excess* capacity and bumping cannot occur. Otherwise the capacity is *scarce*, and bumping can occur if too many units are put on sale.

Since the firm knows which case realizes with certainty, it forces an intuitive restriction $p_C \geq p_1$. Otherwise firms could *intentionally* sell discounted products and later bump class-1 customers (as overflow) for a premium of $p_1 - p_C > 0$.

5.2 Single-Period Solution for the Simplified Models

Let $\hat{x}(\alpha, y)$ be the maximum number of discounted units such that: (i) all class-2 customers are allocated a product without bumping others; and (ii) all $\hat{x}(\alpha, y)$ units are sold. In the case of excess capacity the firm cannot sell all available inventory, and so $\hat{x}(\alpha, y) = A(\alpha, y)$. For the scarce capacity case, solving $\alpha y d_2 \left(1 - \frac{x}{A(\alpha, y)} \right) = (N - S - x)$ yields $\hat{x} = A(\alpha, y) \frac{N - S - \alpha y d_2}{A(\alpha, y) - \alpha y d_2} < A(\alpha, y)$. Observe $\hat{x} \leq N - S$. In summary,

$$\hat{x}(\alpha, y) = \begin{cases} A(\alpha, y), & \text{if } A \leq N - S \text{ (excess capacity case);} \\ A(\alpha, y) \frac{N - S - \alpha y d_2}{A(\alpha, y) - \alpha y d_2}, & \text{if } A > N - S \text{ (scarce capacity case).} \end{cases} \quad (10)$$

Theorem 3 *For a single period problem, in either simplified model, there exists a threshold α^* , such that if $\alpha \geq \alpha^*$ then $x^* = 0$. Otherwise, in the case of scarce capacity, $x^* = \hat{x}(\alpha, y)$, and in the case of excess capacity any $x \in [\hat{x}(\alpha, y), N - S]$ is optimal.*

We note that $\alpha^* \leq 1$ if $D_t(Y_t)p_1 \leq Y_t d_2 p_2$. That is, $\alpha^* \leq 1$ if the potential revenue at price p_2 exceeds the potential revenue at p_1 . In this case, the single-period optimal policy

is “bang-bang”: the optimal number of discounted units drops down to zero if too many customers wait (i.e., when $\alpha \geq \alpha^*$); and it jumps up to \hat{x} otherwise. In the opposite case, if $\alpha^* > 1$ then α is always smaller than α^* and some number of units are always placed on a discount. Next we prove that a similar “bang-bang” policy holds for every period.

5.3 Multiple-Period Solution for Simplified Models

Let $R_t(\theta_t, \alpha_t, y_t, x_t)$ be the expected revenue-to-go, given that for period t , the waiting parameter is θ_t , the realized waiting fraction is α_t , the observed demand multiplier is y_t and x_t units are put on sale. The optimal number of discounted units, x_t^* , can be found for each period, $t = 1, 2, \dots, T$, by solving the following dynamic program:

$$R_t(\theta_t, \alpha_t, y_t, x_t) = g_t(\alpha_t, y_t, x_t) + \delta E_{(\alpha_{t+1}, y_{t+1}) | \theta_{t+1} = h_t(\theta_t, x_t)} [R_{t+1}^*(\theta_{t+1}, \alpha_{t+1}, y_{t+1})] \quad (11)$$

where the optimal revenue-to-go is given by

$$R_t^*(\theta_t, \alpha_t, y_t) = \max_{0 \leq x_t \leq N - (1 - \alpha_t)y_t d_2} R_t(\theta_t, \alpha_t, y_t, x_t) \quad (12)$$

where $g_t(\alpha_t, y_t, x_t)$ is given by (9); $R_{T+1}^*(\theta_{T+1}, \alpha_{T+1}, y_{T+1}) = 0$ for all $(\theta_{T+1}, \alpha_{T+1}, y_{T+1})$; and $\theta_{t+1} = h_t(\theta_t, x_t)$, is increasing and concave in either argument. Because $\theta_{t+1} = h_t(\theta_t, x_t)$ does not depend on the realized values of α_t and y_t , we can write $R_t(\theta_t, \alpha_t, y_t, x_t) = g_t(\alpha_t, y_t, x_t) + \delta \phi_{t+1}(h_t(\theta_t, x_t))$. In our two simplified models ϕ_{t+1} takes the following specific forms:

- *In the deterministic waiting fraction (DW) model*, $\alpha_t \equiv \theta_t$ for all t by assumption and the distribution of Y_t is independent of α_t . Therefore

$$\phi_{t+1}^{DW}(h_t(\alpha_t, x_t)) = E_{y_{t+1}} [R_{t+1}^*(h_t(\alpha_t, x_t), y_{t+1})] \quad (13)$$

- *In the deterministic demand (DD) model*, $Y_t \equiv 1$ for all t by assumption, and so w.l.o.g. y can be dropped from the expectation of the future revenue, leading to

$$\phi_{t+1}^{DD}(h_t(\theta_t, x_t)) = E_{(\theta_{t+1}, \alpha_{t+1}) | \theta_{t+1} = h_t(\theta_t, x_t)} [R_{t+1}^*(\theta_{t+1}, \alpha_{t+1})] \quad (14)$$

Let $\Pi_t = \{x_t : R_t(\theta_t, \alpha_t, y_t, x_t) = R_t^*(\theta_t, \alpha_t, y_t) \text{ and } 0 \leq x_t \leq N - S_t\}$ be the set of “potentially optimal” solutions for period t . Our main result for the simplified models is that $\Pi_t = \{0; \hat{x}_t\}$ for all $t = 1, 2, \dots, T$. The concept of our proof is the following. Suppose that the expected future revenue, ϕ_{t+1} , is decreasing and convex in x_t and α_t . Since $g_t(\alpha_t, x_t, y_t)$ is piecewise linear in x , R_t consists of two adjacent and convex segments. Since g is also

decreasing for $x_t > \hat{x}_t$, R_t is also decreasing if $x_t > \hat{x}_t$. Therefore $x_t^* \in \{0; \hat{x}_t\} \equiv \Pi_t$. We summarize this result in the theorem below.

Theorem 4 *In either simplified model, $x_t^* \in \Pi_t = \{0; \hat{x}_t\}$ for all periods $t = 1, 2, \dots, T$.*

To summarize, for the case with a smoothing learning function for both simplified models, the optimal policy is “bang-bang”; it places either 0 or \hat{x}_t units on the end-of-period sale depending on the realized waiting fraction, α_t . The firm increases the number of “shoppers” by offering units on sale, and then withdraws revenue from those shoppers by periodically not offering any discounted units, subsequently decreasing the likelihood of future waiting.

By following such policy the firm simultaneously achieves high utilization of its capacity and controls the number of customers waiting. This policy is quite different from that of the self-regulating case, because the firm actively manages the waiting, as opposed to relying on the consumers to regulate the waiting themselves. Observe that since $\hat{x}_t \leq N - S_t$ bumping is never optimal. This is because the marginal revenue p_2 per unit could as well be obtained from the initial sales, and since the future revenue is decreasing in x_t , the firm puts fewer units on sale and reduces the future waiting. This is not the case if the firm can obtain a marginal revenue in excess of p_2 , as happens in the three price model that we study next.

6 Three Price Model

Next we study the case where a firm may choose to offer some units for sale at p_1 , while raising the price to a higher value for the remaining inventory. By doing so the firm can both capture the low-price demand, as well as the demand willing to pay extra for being accommodated after all discounted units are already sold. The three-price model reflects frequently observed situations where the “walk-up” price is higher than the regular, while some units have been sold at a discount earlier.

Let $p_3 \geq p_2$ be the “high” price. We build up on the simplified model of Section 5.1 amended as follows. We assume only class-3 customers are willing to purchase at price p_3 and let their demand in period t be $Y_t d_3 \equiv Y_t D_3$. The number of customers who are willing to pay price p_2 is now $Y_t D_2$ where $D_2 = d_2 + d_3$. Also let $D_1 = d_1 + d_2 + d_3$. We assume $\bar{y} D_2 < N$ as before. At the sale time the firm decides x_t , the number of units to offer at the discounted price p_1 . The remaining units are offered at price p_3 .

To determine the revenue of the firm, $S_t \equiv (1 - \alpha_t)Y_t D_2$ units are sold at the initial price p_2 , and $\min[x_t, A_t]$ units are sold at the discounted price p_1 , where $A_t \equiv (a + bY_t)d_1 + \alpha_t Y_t D_2$. Let $\psi_t(\alpha_t) \in [0, 1]$ be the fraction of class-3 customers who wait for a discount, given that there is a fraction $\alpha_t \in [\underline{\alpha}; \bar{\alpha}]$ of class-2 and -3 customers waiting combined. That is, the total number of class-3 customers waiting is $\psi_t(\alpha_t)Y_t d_3$, and the number of class-2 customers is $\alpha Y_t D_2 - \psi_t(\alpha_t)Y_t d_3$. Since the latter is non-negative, it is implied that $\psi_t(\alpha_t)Y_t d_3 \leq \alpha Y_t D_2$ for all $\alpha_t \in [\underline{\alpha}; \bar{\alpha}]$.

As before, we assume that the discounted units are allocated on proportion. That is, the number of discounted units that are sold to class-3 customers is $\min[x_t, A_t] \frac{\psi_t(\alpha_t)Y_t d_3}{A_t}$ and the net single-period revenue is

$$g_t(\alpha_t, Y_t, x_t) = p_2 S_t + p_1 \min[x_t, A_t] + p_3 \psi_t(\alpha_t) Y_t d_3 \left(1 - \frac{\min[x_t, A_t]}{A_t} \right) - p_C \left(\psi_t(\alpha_t) Y_t d_3 \left(1 - \frac{\min[x_t, A_t]}{A_t} \right) - (N - S_t - x_t) \right)^+,$$

where $p_1 \leq p_C \leq p_3$.

In this section we assume that the demand multiplier, Y_t , is stochastic, and the waiting fraction, α_t , is deterministic.[‡] We assume that the distribution of Y_t does not depend on α_t , and that in turn, α_t evolves according to a linear learning function $\alpha_{t+1} = h_t(\alpha_t, x_t)$. We place no restriction whether $h(\cdot)$ is smoothing or self-regulating.

In order to proceed we need to further specify the properties of $\psi(\alpha)$. Consider the case where it is the firm's policy not to oversell capacity and subsequently lose unsatisfied customer demand (the "no-bumping" case). Because class-3 customers have a higher valuation for the product, they are less willing to wait and risk not receiving a unit. Therefore we assume ψ is "small," compared with α , and class-3 customers do not wait unless many class-2 customers are already waiting. For example, if we assume that class-3 customers do not wait, unless *all* class-2 customers are already waiting, then $\psi(\alpha) = \max[0, 1 - \frac{D_2}{d_3} + \alpha \frac{D_2}{d_3}]$ for $\alpha \in [0, 1]$.

In the case where the firm is willing to bump passengers (the "bumping" case), class-3 customers who waited but did not get a discounted unit do not have capacity concerns, as they are guaranteed a product at their reservation price, p_3 (since $y_t d_3 \leq \bar{y} D_2 \leq N$).

[‡]Alternative models with stochastic demand and waiting, or deterministic Y and stochastic α , could in general be of interest as well. However, we found that in three price context they place restrictive and uninterpretable conditions on prices, demands and waiting. Therefore we do not present them.

Therefore, they are more likely to wait, and we assume ψ is “large”; there may be a fraction of class-3 customers waiting even if no class-2 customers wait. For example if $\psi(\alpha) = \frac{d_1}{D_1} + \alpha \frac{D_2}{D_1}$ for $\alpha \in [\frac{D_3(D_1-D_2)}{D_2(D_1-D_3)}, 1]$, the fraction of class-3 customers that always wait is $\psi(\underline{\alpha}) = d_1/(d_1 + d_2)$.

We assume that ψ is nondecreasing convex. In the no-bumping case $\psi = 0$ for $\alpha \leq \hat{\alpha}$ for some $\hat{\alpha} \in [\underline{\alpha}, d_2/D_2]$ and $\psi' D_3 = D_2$, otherwise. In the bumping case, $\psi' D_3 \leq D_2$ and $\psi' A(\alpha, y) \leq \psi y D_2$. Functions that satisfy these assumptions correspond to the “small” and “large” ψ as per the discussion above.

We redefine \hat{x} to include the waiting of class-3 customers as follows:

$$\hat{x}(\alpha, y) = \begin{cases} A(\alpha, y), & \text{if } A \leq N - S \text{ (excess capacity case);} \\ A(\alpha, y) \frac{N-S-\psi(\alpha)yD_3}{A(\alpha,y)-\psi(\alpha)yD_3}, & \text{if } A > N - S \text{ (scarce capacity case).} \end{cases} \quad (15)$$

By the same argument as in the proof of Theorem 3 we obtain that the single-period optimal policy resembles that of the two-price model.

Theorem 5 *In the three price model there exists a threshold α^* , such that if $\alpha \leq \alpha^*$, then $x^* = \hat{x}(\alpha, y)$ in the case of scarce capacity, and any $x \in [\hat{x}(\alpha, y), N - S]$ is optimal in the case of excess capacity. Otherwise, $x^* = 0$.*

In this case, $\alpha^* \leq 1$ if $D_t(Y_t)p_1 \leq Y_t D_3 p_3$. That is, if the revenue from the high-price segment exceeds the revenue from all segments at the low-price, then the single-period optimal policy is “bang-bang.” For multiple periods the result extends as follows:

Theorem 6 *In the three-price model with bumping, $x_t^* \in \{0; \hat{x}_t; N - S_t\}$, and without bumping $x_t^* \in \{0; \hat{x}_t\}$ for all periods $t = 1, 2, \dots, T$.*

In sum, the optimal policy follows a pattern similar to that of the two-class model: increasing the fraction of customers waiting by offering units on sale followed by periods where no units are discounted. By doing so, the firm can withdraw revenue from waiting class-3 customers while decreasing future waiting. There is, however, a major difference between the two- and three- price models. In the former, even if the firm’s policy was to bump customers if overflow occurs, doing so was never optimal. In contrast, in the latter, the optimal solution could involve selling all of the available $N - S$ units at p_1 , bumping some customers and paying penalty $p_C < p_3$. By doing so the firm can encourage more class-3 customers to wait, and potentially get revenue p_3 in excess of the regular price p_2 per unit, even though it may result in paying bumping penalties.

7 Numerical Studies

In this section we provide several examples to illustrate the value of making decisions optimally as compared with several heuristics managers use in real-life situations, and examine how this value and the optimal policy itself change in different situations. We also analyze how to select the optimal discount price p_1^* . We use the three price model since it allows for all types of consumer behavior that we study in our paper. We consider the case with positively correlated demands (i.e., with $a = 0, b = 1$ leading to the class-1 demand of $Y_t d_1$).

We set $N = 100$, $\delta = 0.95$, $D_2 = 50$, $p_1 = 100$, $p_2 = 300$ and $p_3 = 500$, and consider four families of instances: smoothing bumping (MB), smoothing no-bumping (MN), self-regulating bumping (RB) and self-regulating no-bumping (RN). For each family of instances we study four demand curves, with $D_1 = 150$ or $D_1 = 100$, and $D_3 = 30$ or $D_3 = 10$ for class-3 customers. We denote these demand curves as “150-50-30,” “150-50-10,” “100-50-30,” and “100-50-10,” respectively. For cases with bumping we study two penalties: $p_C = 150$ and $p_C = 450$.

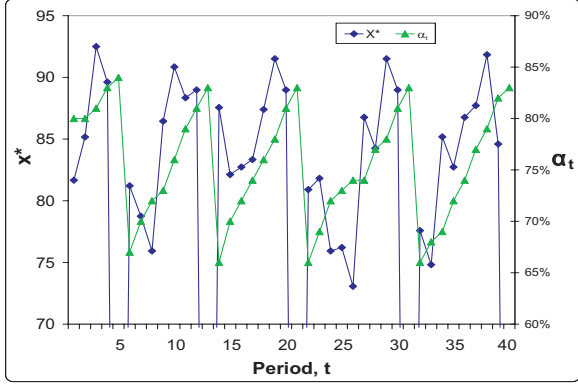
We use functions $h(\alpha, x) = \lambda \frac{x}{N} + (1 - \lambda)\alpha$ and $h(\alpha, x) = (1 - \lambda) + \lambda \frac{x}{N} - (1 - \lambda)\alpha$, for $0 < \lambda < 1$ for the smoothing and self-regulating learning, respectively. We use $\psi(\alpha) = \frac{d_1}{D_1} + \alpha \frac{D_2}{D_1}$ and $\psi(\alpha) = \max[0; 1 - \frac{D_2}{D_3} + \alpha \frac{D_2}{D_3}]$ for waiting functions with and without bumping, respectively.

We set $\underline{y} = 0.6$ and $\bar{y} = 1.4$, so that the demand multiplier $Y_t \in [0.6, 1.4]$, and examine three distributions: truncated *Normal*[1, 0.2][§], *Beta*[1.75, 3] and *Beta*[0.8, 2], with the expected values and CVs, respectively, (1, 0.913, 0.841) and (0.181, 0.191, 0.237). The results are very similar and below we report only the case with *Beta*(0.8;2).

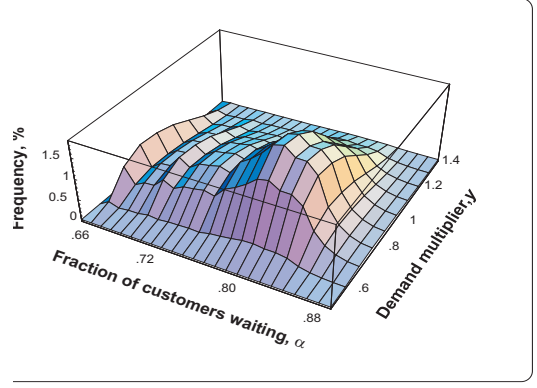
For each instance we compute the expected infinite horizon discounted revenue (the *revenue*), assuming that the system starts from steady state – intuitively, this is the revenue that the firm will generate starting at an arbitrary time in the future. To compute the revenue we discretize α_t as $\{0; 0.01; 0.02; \dots 1\}$ and discretize y_t as $\{0.6; 0.7; \dots 1.4\}$ for a total of 909 states. We use successive approximations with error bounds to compute the infinite horizon expected revenue function value (Bertsekas (1987), pp. 188-193). Then we determine the subset of recurrent states and the steady state probabilities, and obtain the expected revenue as the weighted sum (Puterman 1994, pp.589-594).

In Figure 2 (a) we present a sample path for the optimal decision, x^* , and the fraction

[§]We use 2σ limits so that the endpoints have non-negligible probabilities.



(a)



(b)

Figure 2: In (a): sample path of the optimal decision, x_t^* , and the fraction of customers waiting, α_t . In (b): frequencies of visiting different states by the optimal policy. Both examples are for the MN instance with 150-50-30 demand curve and Beta(0.8,2) demand multipliers; $\lambda = 0.2$.

of customers waiting, α_t , and in (b) we present the recurrent states and the frequencies with which they are visited in the steady state. We observe that the optimal decision and the fraction of customers waiting follow the cycles of variable length. This expresses the “bang-bang” structure of the optimal policy: if in period t the fraction of waiting customers, α_t , gets large enough, then the firm puts $x_t^* = 0$ units on sale and so α_{t+1} drops. Hence in period $t + 1$ the firm puts $\hat{x}(\alpha_{t+1}, y_{t+1}) > 0$ units on sale and α_{t+2} increases again; note the 1-period time lag between x and α in (a). Following such cycles, the optimal policy visits a variety of states; see (b). Because of the uncertainty in demand, the length of cycles is random. Thus, the customers cannot anticipate the transitions and hence the decision of the firm.

7.1 Performance of the Optimal Policy and Managerial Insights

To better understand the performance of the optimal policy we compare it with four heuristics that appeal to managers. In each heuristic we determine the number of units to put on sale through different methods. We consider the following:

Do-nothing: Let $x_t = N - S_t$ or $x_t = 0$ for all t , whichever is better;

BestP: Let $x_t = N - S_t$ with probability P^* and $x_t = 0$ with probability $1 - P^*$, where the value of P^* is the one that results in the highest revenue;

S*: Let $x_t = N - S$ if $S_t \leq S^*$ and $x_t = 0$, otherwise, where the value of S^* is the one that results in the highest revenue;

Beta*: Let $x_t = N - S_t$ with probability $\beta^* \frac{N-S_t}{N}$, and $x_t = 0$ with probability $1 - \beta^* \frac{N-S_t}{N}$, where the value of β^* is the one that results in the highest revenue.

The rationale behind the do-nothing heuristic is straightforward: choose the better of “all” or “none” on sale ad do so consistently in all periods. The BestP heuristic attempts to prevent consumers from guessing if a sale will occur in a given period.[¶] Heuristics S* and Beta* represent a naïve managerial approach where discounts are offered in periods with low regular price sales (i.e., when S_t is small). The former heuristic does so when a threshold is crossed, whereas the latter places units on sale based on a linear probabilistic rule. We find P^* , S^* and β^* through numerical search. Given the revenues of the optimal and heuristic policies, we compute the *relative improvement* of the optimal policy over a particular heuristic as $(\text{optimal revenue} - \text{heuristic revenue}) \div (\text{heuristic revenue})$.

Figure 3 presents the relative improvements over the heuristics for the four families of instances with 150-50-30 demand curve and Beta(0.8;2) demand multipliers. Our main observation is that in all cases the optimal policy generates five to fifteen percent additional revenue over the best heuristic. This value changes depending on the speed of learning and the type of consumer behavior. It also depends on which heuristic is the best.

In the cases without bumping, (Figure 3 (a) and (c)), the best heuristic is BestP, as it outperforms heuristics S* and Beta*. At a first glance this might seem slightly counterintuitive, since the latter are based on the intuitive managerial approach to put more units on sale when S_t is small. However, recall that the optimal policy suggest exactly the opposite to this naïve approach. Specifically, $x_t^* = 0$ if $\alpha_t > \alpha^*$, and since $S_t = (1 - \alpha_t)y_t D_2$, it follows that (in expectation) it is optimal *not* to put units on sale in the periods with small S_t .

The improvement over the BestP heuristic depends on the speed of learning. This is because the optimal policy determines when to offer a discount (the timing), and if one is offered, then how many units to discount (the number). By choosing the best probability, the BestP heuristic “optimizes” the long-run average number of units on sale, but cannot

[¶]A variation of the BestP heuristic is used by a car rental company with whom we discussed our work: they have a deal almost every week, but to access it, customers need a promotion code. These codes are e-mailed to a subset of their registered webmail customers, where a customer is included on the mailing list for a given week with some probability.

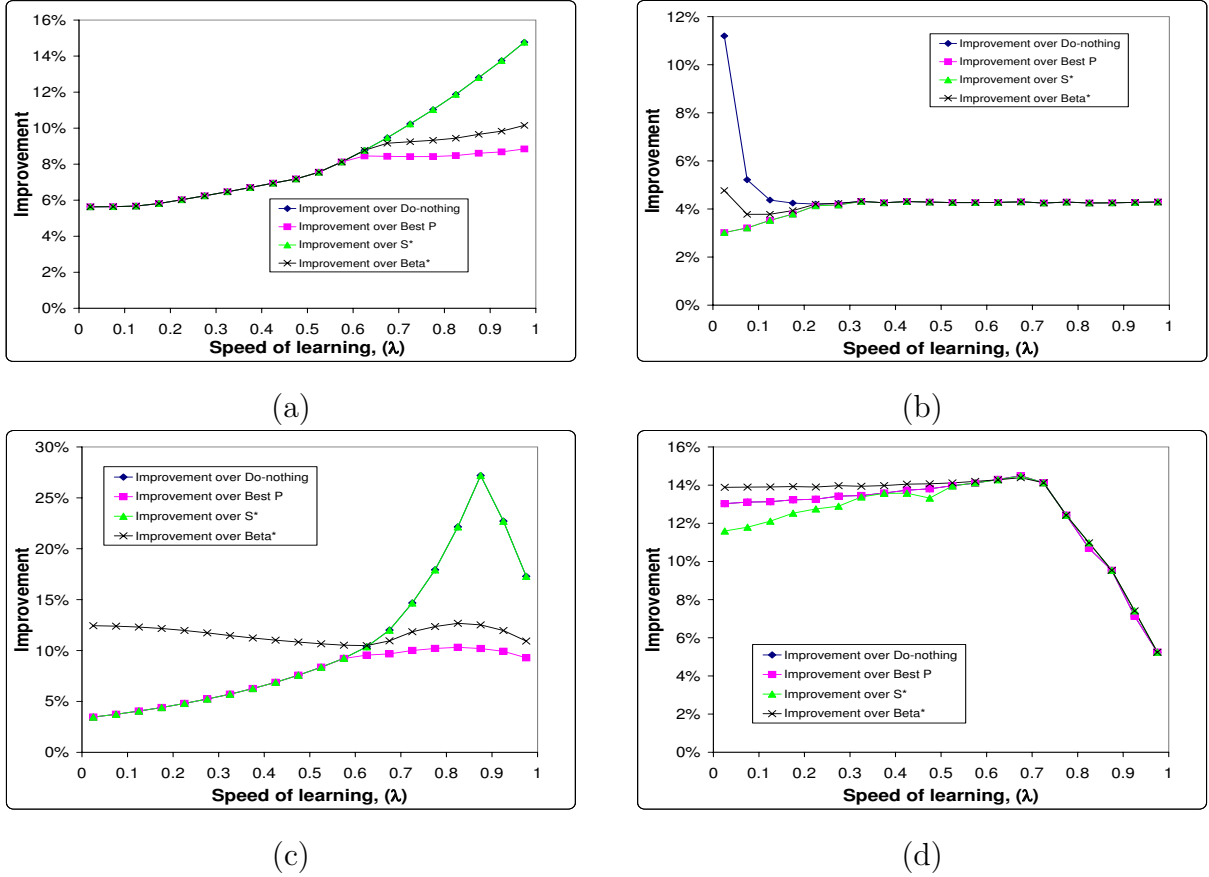


Figure 3: The relative improvements for the (a) MN , (b) MB, (c) RN and (d) RB instances with 150-50-30 demand curve and Beta(0.8,2) demand multipliers. In (b) $p_C = 150$ and in (d) $p_C = 450$.

achieve the right timing of sales. In the cases of slow learning in order to change waiting behavior, the firm must have consistent series of periods with and without discounts. The BestP heuristic cannot ensure such consistency, and therefore chooses to do nothing (indeed $P^* = 0$ for $\lambda < 0.625$ on Figure 3 (a) and for $\lambda < 0.575$ on Figure 3 (c)). For faster speeds of learning, consistency is not required as customers readily change their waiting behavior; therefore the timing of sales is less important than the average number of units on sale.

In the instances with bumping (Figure 3 (b) and (d)), the best heuristic is S*. This is because even though all heuristics bump customers in the situations where the optimal policy does not, S* bumps the least. As discussed above, the S* heuristic has a direct control over the timing, as opposed to the probabilistic heuristics which do not. Better timing implies less frequent bumping and consequently lower (total) bumping penalty, so the S* heuristic returns higher revenue in the cases with bumping than other heuristics. Also, because the total penalty is proportional to p_C , the improvement is naturally larger

Instances where discounts improve revenue	Instances where discounts do no improve revenue
MB, RB 150-50-10	MN, RN 150-50-10
MB, RB 100-50-10	MN, RN 100-50-10
MN, RN 100-50-30	MB, RB 100-50-30
MN, MB, RN, RB 150-50-30	

Table 1: Comparison of cases where end-of-period discounts are beneficial.

in the cases with large p_C : compare (b) and (d) in Figure 3.

Next we study the factors that influence whether strategic revenue management as we discuss in this paper will be effective. Table 1 classifies different instances into those where the firm benefits from offering end-of-period discounts and those where it does not. In the cases with few class-3 customers (rows 1 and 2 in Table 1) observe that discounts increase revenue only in the cases with bumping. This is because if the firm bumps customers to accommodate class-3 overflow, then it provides an incentive for more class-3 customers to wait (through the $\psi(\alpha)$ function). But because there are few class-3 customers overall, the number of waiting class-3 customers is still small compared to the number of class-1 customers. As a result, only a few class-3 customers who wait are able to buy at a discount, while most buy at p_3 , hence increasing firm's revenue. In contrast, when there are many class-3 customers and few class-1 customers (row 3), the threat imposed by refusing to bump customers forces fewer high-value class-3 customers to wait (through $\psi(\alpha)$), allowing the firm to sell much of its capacity early at price p_2 and gain revenue by offering discounts that are mostly purchased by class-1 customers. Without the threat (i.e., with bumping) the reaction function $\psi(\alpha)$ changes and too many class-3 customers wait and purchase discount seats, instead of purchasing them early at price p_2 . Thus, given a choice, depending on the relative class-1 and class-3 demand, the firm may or may not prefer to oversell its capacity and bump passengers. That is, the firm can use bumping strategically to induce appropriate customer behavior and enhance the effectiveness of its revenue management policy.

7.2 Selecting the Optimal Discount Price, p_1^*

Our model assumes that the discount price, p_1 , is fixed for the entire horizon of T periods, and as we argue in the introduction, building a model where customers react to both price and availability of discounted units is nontrivial. As research in dynamic pricing shows, however, a heuristic that charges an optimally selected single price (as opposed to optimizing

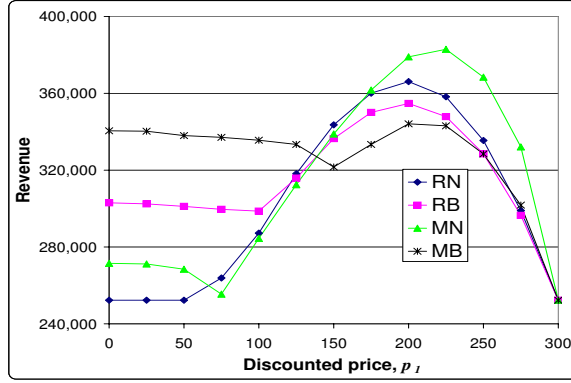


Figure 4: Optimal revenue as a function of discounted price, p_1 , in the instances with Beta(0.8,2) demand multipliers.

it dynamically) often performs only marginally suboptimally. Therefore, as a heuristic policy, the firm could search for the optimal “static” discounted price, p_1^* , charge it in every period, and then determine the number of discounted units to put on sale following our optimal policy.

We search for such optimal static price, p_1^* , numerically over its domain, $[0, p_2]$; see Figure 4. In this example, we impose a demand curve $D(p) = 200 - 0.5p$ (so that $D(300) = 50$ and $D(100) = 150$ as in previous examples). We also assume that the value of a discount influences the rate at which customers are willing to change their behavior, i.e., the speed of learning. In particular we assume $\lambda(p) = 0.3 - 0.001p$. We experimented with other functions for demand and speed of learning, but observed no qualitative differences from the case presented, see Figure 4.

Two observations are evident from Figure 4. First, the revenue is not concave and often not quasiconcave in the price, therefore it may not be possible to determine the optimal discount analytically. There is an obvious optimal point, however. Such a point exists because the firm uses class-1 demand to achieve two goals. On one hand it wants p_1 to be high, since then it obtains larger revenue from the sale of each discounted unit. On the other hand, it wants $D(p_1)$ to be high, because under proportional allocation class-1 customers displace some waiting class-3 customers, so that they purchase at the higher price, p_3 . Since price negatively affects demand, these goals conflict and an optimal trade-off point is found.

Figure 4 also provides a neat illustration to the earlier point that firms can strategically use bumping to increase revenue. Observe that when the discounted price is small, hence, class-1 demand is high, more revenue is obtained when the firm bumps passengers (RB

and MB curves are higher than RN and MN). Conversely, when the price is large, class-1 demand is small, and the firm benefits from not bumping (RN and MN curves are higher).

8 Conclusions

Our work is motivated by the concern that given the increased ability to search for better prices for travel related products (flights, vacation packages, etc.), consumers will learn to expect end-of-period deals and will strategically wait for them. We study the problem for cases of two and three customer classes. The two-class problem represents the case where a list price is given (as in the cruise or vacation packages industries); the three-class problem reflects typical airline pricing where prices may decrease or increase in the days prior to departure. We formulate the problem as a dynamic program and develop a unique solution approach amenable to the novel structural properties we find in the problem.

For the case of two customer classes with self-regulating customer behavior, we show that the firm in general will set some units on sale in each period and allow the customer behavior to limit the number receiving the benefit of the reduced price inventory. In contrast, in the case of smoothing customer behavior, we show the firm should follow a “bang-bang” sale policy, either placing most of the remaining units on sale or none. Thus, the firm takes a more active role, adjusting the customers’ expectations by alternately increasing the number of customers waiting until a threshold is crossed, upon which the firm places no units on sale. By doing so, the firm is able to regulate the number of customers waiting and to increase its revenue by increasing utilization, allowing some units that would otherwise not be sold to be purchased by the lower-value customers.

In the model with three customer classes, we consider how high-value customers react to a firm’s bumping policy. In the cases with bumping, high-value customers have a higher incentive to wait. As a result, in addition to the “bang-bang” policy, it could be optimal to discount all units remaining and pay bumping penalty. Such policy, however, is beneficial only if there are few high-value customers. Establishing a policy of not bumping passengers is beneficial when there are many high-value customers. Overall, the benefit from following the optimal policy is 5 to 15 percent more revenue as compared to several intuitive managerial heuristics.

We acknowledge that the model formulated here does not account for all factors that

may influence strategic customer behavior in revenue management. Future studies should consider more explicit formulations of customer utility, competition between firms and gaming by both firms and customers. In addition, empirical work is needed to better understand consumer learning behavior with regards to travel-related discounts.

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Appendix A: Model Discussion

Aggregate Demand. We consider a model where rather than tracking the detailed arrival dynamics of individual customers, the firm focuses on the aggregate behavior of customer classes through the parameter θ . Our approach is motivated by many discussions with revenue management executives who noted that in a multiple-period setting like ours, customers who purchase products in different periods are typically different individuals, and it is rather unclear how these individuals react to the firm's revenue management policy or even if they have accurate information about it at all. At the same time, the firm is concerned with the aggregate outcome of these individual behaviors, and as these executives noted, there exists information sources such as industry reports, news articles, and web sites, through which aggregate demand reacts to the revenue management policy of the firm. Our model expresses such considerations.

In our general model we place no restrictions on how class-1 and -2 demands combine in the total demand D_t , and allow the demands from different classes to be arbitrarily positively or negatively correlated, or be independent. This is important because such cases could realistically arise depending on the interpretation of demand parameter Y_t . For example, weather is likely to influence both classes in the same direction (resulting in a positive correlation), but exchange rate could lead to a shift in market segments – some discount only class-1 customers could become regular price class-2 customers if exchange rate improves (resulting in a negative correlation). In our simplified models, Sections 5 and 6, we consider the case where the demand in class i reflects a nominal demand d_i and a random multiplier. In class -2 (and -3) this nominal demand is multiplied by the demand parameter Y_t , and in class-1 the nominal demand is multiplied by a linear function of Y_t , $a + bY_t$. Then the cases with $b > 0$ and $b < 0$ reflect positive and negative (although perfect, up to a linear transformation) correlations, respectively. Appendix C presents an extension of simplified models to the non-perfectly correlated demands.

Waiting Parameter and Waiting Fraction. Our key differentiating assumption from previous work is that the fraction of class-2 customers who wait is governed by a parameter, θ_t , that changes with customer learning. Among the factors that lead to an environment where some customers wait and some buy early are anxiety/risk and anticipation. Customers who wait for an end-of-period deal may experience anxiety and risk because, as noted above,

they are not guaranteed a product. Researchers in marketing (e.g., Nowlis et. al. 2004) and economics (e.g., Loewenstein 1987) showed that for “pleasure” products, of which a vacation is an archetypal example, there exists a positive utility of anticipation. For a constant price, customers who purchase a product earlier obtain a higher utility from consuming it. We note that the utility of anticipation, widely recognized by executives in the vacation industry as one of the drivers of early purchases, creates another tradeoff against waiting. Heterogeneity of the customer population with respect to valuing such time and risk tradeoffs (Chesson and Viscussi 2000) naturally yields the aggregate outcome that some customers wait and some purchase early, which is observed in practice and is reflected in our model through waiting parameter θ_t .

Bumping as a result of overselling capacity is a common practice in travel industry. Customers are well aware of the possibility of being bumped, a direct language about overselling/overbooking is printed on the ticket (officially known as the “Contract of Carriage”^{||}) advising that “...*flights may be overbooked, and there is a slight chance that a seat will not be available on a flight for which a person has a confirmed reservation.*” Historically, bumping of passengers was attributed to overestimation of the number of travelers who do not show up for a flight, but no-shows are not mentioned in the contracts we looked at. Rather the contracts state that firms bump passengers “... *in accordance with [its] particular boarding priority*” and bumped customers have no ability to verify if the firm indeed overestimated the no-shows or rather oversold the capacity opportunistically to increase revenue.

There exists significant evidence that firms have full incentive to oversell inventory for higher revenue without considering no-shows. Bialogorsky et al. (1999) and Bell (2008) provide analytical justification for such policies and McCartney (2007) provides anecdotal evidence that ... *overbooking can often be a bit of a windfall for an airline. Sell an expensive last-minute ticket on a full flight to a business traveler and then bump a cheap-fare vacationer, and the airline increases revenue...*”

Bumping in our model reflects such opportunistic overselling of capacity to increase revenue; we do not consider no-shows. We assume that the firm denies boarding to some customers who bought at p_1 and a class-2 customer denied a unit cannot re-book by paying the difference $p_2 - p_1$ (or $p_3 - p_1$ in three price model); this assumption reflects the cur-

^{||}We checked Contracts for 3 major airlines and they are very similar; we do not provide specific references.

rent practice in travel industry** and is reasonable from consumers’ perspective: if a “big spender” buys a cheap ticket hoping for a deal, she should naturally expect to be treated as a low spender.

Overall, because both in practice and in our model firms oversell to increase revenue and bump low-price customers when profitable, bumping clearly has a major impact on consumers’ wait-or-buy behavior. Thus it is important to consider the implications of strategic consumer behavior with and without bumping, as we do in this paper.

Allocation of Discounted Units. In practice, discounted units are sold on a “first come, first serve” basis and the class of the customer is not known. Thus, $B_t(S_t, x_t, Y_t)$ is a result of a random draw in which, for example, all waiting customers could have equal probability of buying a discounted product; then $B_t(\cdot)$ would have a hypergeometric distribution. However, incorporating random allocation leads to an untractable model, in part, because it requires treating the demands and capacities as integers. Therefore, we consider deterministic allocation mechanisms. In Sections 4 and 5, respectively, we discuss two forms of proportional allocation depending on the nominal or realized demands (Talluri and van Ryzin 2004, pp. 330 call such mechanisms “proportional rationing”). Our proportional allocation based on realized demands simplifies $B_t(\cdot)$ to be equal to the expected outcome of the above-mentioned random allocation. In Appendix D, through numerical simulations, we document that the optimal policy obtained with such a simplification is very robust: that is, the revenue generated by such a policy is only marginally different from that resulting from the policy optimal under random allocation.

**We asked managers who deal with bumping in major travel firms to comment on the realism of the non-rebooking assumption. Here are two responses we received:

Respondent 1 “A person in the economy cabin [who bought the lower fare] would not be able to pay more money to be the person not denied boarding in an oversold situation. ... [S/he] will not have the opportunity to ‘jump ahead’ of the other passengers by paying more or upgrading.”

Respondent 2 “In this age of fewer employees working harder than ever before, gate agents are generally solely focused on the operation—that is, getting the flight out on time. That means they will almost always decline to do any ticket transactions at the gate. ... 90-95% of the time, the ‘rich’ customer who asks the agent to upgrade his ticket to get a higher [boarding] priority will likely be told no.”

A third respondent verbally confirmed these statements. All three managers also confirmed that the consumers who paid lowest fares are the first candidates for denying boarding.

Fixed Discounted Price p_1 . In general, firms could determine both the quantity to discount and the sale price for each period. Analyzing both variables simultaneously, however, leads to a very complex multi-period model, since doing so requires understanding the customer’s perception of how quantity complements/substitutes for price. For example, we would need to make assumptions on the effect on future customers for the case when the firm offered only few units on sale, but the price was very low, as opposed to the case when the firm put many units on sale, but the discount was small. Therefore, in attempt to create a stylized parsimonious model we assume that the p_i ’s are fixed for the entire T periods and concentrate on the quantity decision. Furthermore, research has shown that a heuristic that charges the properly chosen single price instead of a dynamic price often performs just marginally suboptimally (e.g., Gallego and van Ryzin 1994). In Section 7.2 we demonstrate how one might determine the optimal static discount price while dynamically optimizing the quantity to discount.

Learning Behaviors. Finally, we investigate both self-regulating and smoothing $h(\theta_t, x_t)$ functions, in Sections 4 and 5, respectively. For “smoothing” functions the next period’s waiting parameter, θ_{t+1} , lies between θ_t and x_t/N , so that the decision x_t is “smoothed” into the previous belief, θ_t . Smoothing functions represent the standard moving average forecasting and are frequently used to describe the dynamics of the aggregate demand, e.g., Greenleaf (1995), Popescu and Wu (2007), Liu and van Ryzin (2009). Alternately, “self-regulating” functions reflect the following behavior: as the total number of waiting customers increases, the chances to obtain a product on sale decrease for an individual customer. This negatively affects the number of customers waiting – an expression of self-regulation captured in $h(\theta_t, x_t)$.

Recall that learning in our model applies to the aggregate demand and not to the individual consumers. Since the update of waiting fraction in our model happens ex post after period t is over, at the aggregate level there are many sources of information about the outcome of period t : word of mouth, advertisements, web sites, surveys, industry reports, personal experiences, and alike. It is, therefore, quite plausible to assume that the aggregate demand behavior changes with x_t ; it is certainly *much more* plausible than an alternate assumption made in most other works that each and every individual’s behavior changes in an identical way, either because they are all in the market every period, or including those not in the market in a given period.

Appendix B: Proofs

We require the following fundamental theorem:

Theorem 7 (3.9.1 in Topkis 1998) *If T is a subset of R^m , $\{F_X(x; \theta) : \theta \in T\}$ is a collection of distribution functions, and \mathcal{F} is a closed (in the topology of pointwise convergence), convex cone of real-valued functions on T , then for any increasing set S , $\int_S dF_X(x; \theta)$ is in \mathcal{F} iff $\int_S v(x) dF_X(x; \theta)$ is in \mathcal{F} for any increasing real-valued function $v(x)$.*

We use this Theorem to establish monotonicity properties of expectations of functions of random variables. For example, if $v(x)$ is an increasing function in x , and X is a random variable that is stochastically concave in some parameter θ , (recall that this means that cdf of X is convex in θ), then by Theorem 7 the expectation of $v(x)$ over x , $E_X[v(x)]$, is concave in θ . Similarly, if v is decreasing in x , and X is stochastically convex in θ , then $w = -v$ is increasing (in x), hence the expectation of w is convex (in θ), i.e., the expectation of v is concave (in θ), and so on.

We also require the following two results:

Proposition 1 (follows from Heyman and Sobel, (1984), p. 525) ^{††} *If X is a convex set, $Y(x)$ is a nonempty set for every $x \in X$, the set $C = \{(x, y) | x \in X, y \in Y(x)\}$ is a convex set, $g(x, y)$ is a concave function on C ,*

$$f(x) = \sup_{y \in Y(x)} g(x, y)$$

and $f(x) < \infty$ for every $x \in X$, then f is a concave function on X .

Note that the above Proposition implies that if $f(x, \theta)$ is jointly concave in (x, θ) on a convex subset of R^2 , then $\sup_x f(x, \theta)$ is concave in θ .

Proposition 2 (Theorem 2.7.6 in Topkis 1998) *If $f(x, \theta)$ is supermodular in (x, θ) on a convex subset of R^2 , then $\sup_x f(x, \theta)$ is supermodular in θ .*

Proof of Lemma 1. Recall that by assumption (A5) h is linear.

(i) Concavity follows from

$$\frac{\partial^2}{\partial x^2} J(\theta, S, x) = \frac{\partial^2}{\partial x^2} r(\theta, S, x) + \delta \frac{\partial^2 \phi}{\partial h^2} \left(\frac{\partial h}{\partial x} \right)^2 \leq 0$$

^{††}See also Theorem A.4 in Porteus (2002) p. 227, and Theorem 5.3 in Rockefellar (1997) for slightly different statements of effectively the same result.

since r is concave in x by assumption (A1) and ϕ is concave in h by the condition of the lemma.

(ii) Supermodularity in (θ, x) follows from

$$\frac{\partial^2}{\partial x \partial \theta} J(\theta, S, x) = \frac{\partial^2}{\partial x \partial \theta} r(\theta, S, x) + \delta \frac{\partial^2 \phi}{\partial h^2} \frac{\partial h}{\partial x} \frac{\partial h}{\partial \theta} \geq 0$$

since r is supermodular in (θ, x) by assumption (A2), ϕ is concave in h by the condition of the lemma and h is self-regulating by assumption (A5).

(iii) Supermodularity in (θ, S) and (S, x) follows from (A2) because ϕ does not depend on S . Supermodularity in multiple dimensions is equivalent to supermodularity in each pair (Topkis (1998), Theorem 2.6.1). ■

Proof of Lemma 2. Let $\hat{\theta}$ be an arbitrary fixed value of θ . Then

$$\begin{aligned} & \frac{\partial^2}{\partial \theta^2} \left(\int v(\theta, x) dF_X(x, \theta) \right) \Big|_{\theta=\hat{\theta}} = \frac{\partial}{\partial \theta} \left(\int \left(\frac{\partial}{\partial \theta} v(\theta, x) \right) dF_X(x; \theta) + \int v(\theta, x) d \frac{\partial}{\partial \theta} F_X(x; \theta) \right) \Big|_{\theta=\hat{\theta}} \\ & = \int \left(\frac{\partial^2}{\partial \theta^2} v(\theta, x) \right) \Big|_{\theta=\hat{\theta}} dF_X(x; \hat{\theta}) + 2 \int \left(\frac{\partial}{\partial \theta} v(\theta, x) \right) \left(d \frac{\partial}{\partial \theta} F_X(x; \theta) \right) \Big|_{\theta=\hat{\theta}} \\ & \quad + \int v(\hat{\theta}, x) \left(d \frac{\partial^2}{\partial \theta^2} F_X(x; \theta) \right) \\ & \leq 2 \int \left(\frac{\partial}{\partial \theta} v(\theta, x) \right) \left(d \frac{\partial}{\partial \theta} F_X(x; \theta) \right) \Big|_{\theta=\hat{\theta}} + \int v(\hat{\theta}, x) \left(d \frac{\partial^2}{\partial \theta^2} F_X(x; \theta) \right) \Big|_{\theta=\hat{\theta}} \\ & \leq \int v(\hat{\theta}, x) \left(d \frac{\partial^2}{\partial \theta^2} F_X(x; \theta) \right) \\ & = \frac{\partial^2}{\partial \theta^2} \int v(\hat{\theta}, x) dF_X(x; \theta) \\ & \leq 0 \end{aligned}$$

The first inequality follows from the concavity of $v(\theta, x)$ in θ . The second inequality follows by Theorem 7 because $v(\theta, x)$ is supermodular (i.e. $\partial v / \partial \theta$ is increasing in x), while X_θ is stochastically decreasing. Finally, the third inequality results from Theorem 7 because $v(\theta, x)$ is increasing in x and X_θ is stochastically concave. ■

Proof of Theorem 1. For $t = T$ the claim holds by assumptions (A1) and (A2) respectively. Let $1 \leq t \leq T$ and suppose that for every period $n \in [t+1, T]$: **(I1)** $J_n^*(\theta_n, S_n)$ is increasing in S_n ; **(I2)** $J_n^*(\theta_n, S_n)$ is concave in θ_n and **(I3)** $J_n^*(\theta_n, S_n)$ is supermodular in (θ_n, S_n) .

With assumptions (A4) and (I1)-(I3) following Lemma 2, ϕ_{t+1} is concave in h_t , and therefore with assumptions (A1), (A2) and (A5) by Lemma 1, $J_t(\theta_t, S_t, x_t)$ is concave in x_t and supermodular in (θ_t, S_t, x_t) .

For period t : (I1) follows from (A3) since ϕ_{t+1} does not depend on S_t ; (I2) follows by Proposition 1 from (A1) and because (A5) implies that the Hessian of ϕ_{t+1} equals:

$$\begin{aligned} & \left(\frac{\partial^2 \phi}{\partial h^2} \left(\frac{\partial h}{\partial x} \right)^2 + \frac{\partial \phi}{\partial h} \frac{\partial^2 h}{\partial x^2} \right) \left(\frac{\partial^2 \phi}{\partial h^2} \left(\frac{\partial h}{\partial \theta} \right)^2 + \frac{\partial \phi}{\partial h} \frac{\partial^2 h}{\partial \theta^2} \right) - \left(\frac{\partial^2 \phi}{\partial h^2} \left(\frac{\partial h}{\partial x} \frac{\partial h}{\partial \theta} \right) + \frac{\partial \phi}{\partial h} \frac{\partial^2 h}{\partial \theta \partial x} \right)^2 \\ & = \frac{\partial^2 \phi}{\partial h^2} \left(\frac{\partial h}{\partial x} \right)^2 \frac{\partial^2 \phi}{\partial h^2} \left(\frac{\partial h}{\partial \theta} \right)^2 - \left(\frac{\partial^2 \phi}{\partial h^2} \left(\frac{\partial h}{\partial x} \frac{\partial h}{\partial \theta} \right) \right)^2 = 0. \end{aligned}$$

(I3) follows by Proposition 2 since $J_t(\theta_t, S_t, x_t)$ is supermodular in (θ_t, S_t, x_t) . ■

Proof of Lemma 3. Recall that $y_L(S, x) = \{y : D(y) \leq S + x\}$ and $y_H(S, x) = \{y : B(S, x, y) > N - S - x\}$; note that by (B4), $y_L(S, x) \cap y_H(S, x) = \emptyset$.

By the definition of $y_L(S, x)$ and $y_H(S, x)$:

$$g(S, x, \hat{y}) = \begin{cases} p_2 S + p_1 (D(\hat{y}) - S), & \text{if } \hat{y} \in y_L(S, x); \\ p_2 S + p_1 x + p_2 B(S, x, \hat{y}), & \text{if } \hat{y} \notin y_L(S, x) \cup y_H(S, x); \\ p_2 S + p_1 x + (p_2 - p_C) B(S, x, \hat{y}) + (N - S - x) p_C, & \text{if } \hat{y} \in y_H(S, x). \end{cases}$$

By assumption (B1), $g(S, x, \hat{y})$ is non-decreasing in \hat{y} . Similarly, since $p_2 > p_1$, by assumption (B3), $g(S, x, \hat{y})$ is non-decreasing in S .

Let $f_{\hat{Y}}(y; S)$ be the pdf of \hat{Y} , $F_{\hat{Y}}(y; S)$ be the CDF and let \hat{S} be an arbitrary fixed value of S .

Then $r(\theta, S, x)$ is increasing in S because

$$\begin{aligned} & \frac{\partial}{\partial S} \left(\int g(S, x, y) dF_{\hat{Y}}(y; S) \right) \Big|_{S=\hat{S}} \\ & = \int \frac{\partial}{\partial S} (g(S, x, y) f(y; S)) \Big|_{S=\hat{S}} dy \\ & = \int \left(\frac{\partial}{\partial S} g(S, x, y) \right) \Big|_{S=\hat{S}} f_{\hat{Y}}(y, \hat{S}) dy + \int g(\hat{S}, x, y) \left(\frac{\partial}{\partial S} f_{\hat{Y}}(y; S) \right) \Big|_{S=\hat{S}} dy \\ & \geq \int g(\hat{S}, x, y) \left(\frac{\partial}{\partial S} f_{\hat{Y}}(y; S) \right) \Big|_{S=\hat{S}} dy \\ & = \frac{\partial}{\partial S} \int g(\hat{S}, x, y) dF_{\hat{Y}}(y; S) \\ & \geq 0 \end{aligned}$$

The first inequality holds because $g(S, x, \hat{y})$ is nondecreasing in S . The second holds by Theorem 7 because $g(S, x, \hat{y})$ is increasing in \hat{y} while \hat{Y} is stochastically increasing in S .

Similarly, $r(\theta, S, x)$ is concave in θ by Theorem 7 because $g(S, x, \hat{y})$ is nondecreasing in \hat{y} and \hat{y} is stochastically concave in θ . ■

Proof of Lemma 4. Let \hat{x} be the solution to $B(S, \hat{x}, \hat{y}) = N - S - \hat{x}$. Since B is decreasing in x and $\partial B/\partial x \geq -1$, then $B(S, x, \hat{y}) \leq N - S - x$ if $0 \leq x \leq \hat{x}$, and $B(S, x, \hat{y}) > N - S - x$ if $\hat{x} < x \leq N - S$. Because $B(S, x, \hat{y})$ is nonnegative, $\hat{x} \leq N - S$.

Consider two cases:

Case 1: if $D(\hat{y}) > N$ then $x \leq N - S \leq D(\hat{y}) - S$. So from (1) we obtain

$$g(S, x, \hat{y}) = \begin{cases} p_2 S + p_1 x + p_2 B(S, x, \hat{y}), & \text{if } 0 \leq x \leq \hat{x}; \\ (p_2 - p_C)S + (p_1 - p_C)x + (p_2 - p_C)B(S, x, \hat{y}) + p_C N, & \text{if } \hat{x} < x \leq N - S. \end{cases}$$

Noting $g(S, x, \hat{y})$ is continuous at \hat{x} , concave on both segments and $\lim_{x \uparrow \hat{x}} \frac{\partial g}{\partial x} = \lim_{x \downarrow \hat{x}} \frac{\partial g}{\partial x} + p_C(1 - \frac{\partial B}{\partial x})$, $g(S, x, \hat{y})$ is concave by assumption (B2) and $p_C \geq 0$.

Case 2: if $D(\hat{y}) \leq N$ then $S + x + B(S, x, \hat{y}) \leq N$. By (1)

$$g(S, x, \hat{y}) = \begin{cases} p_2 S + p_1 x + p_2 B(S, x, \hat{y}), & \text{if } 0 \leq x \leq D(\hat{y}) - S; \\ p_2 S + p_1(D(\hat{y}) - S), & \text{if } D(\hat{y}) - S < x \leq N - S. \end{cases}$$

$g(S, x, \hat{y})$ is continuous and if $\partial B/\partial x \geq -p_1/p_2$, then $g(S, x, \hat{y})$ is non-decreasing in x , and by assumption (B5) is concave.

Because concavity is maintained under expectation over an exogenous random variable, $r(\theta, S, x)$ is concave in x . ■

Proof of Corollary 1. We use the definitions of $y_L(S, x)$ and $y_H(S, x)$ given in Lemma 3. From (1) and (2) we obtain

$$\begin{aligned} r(\theta, S, x) &= \int_{y_L(S, x)} (p_2 S + p_1(D(y) - S)) dF_{\hat{Y}}(y) \\ &+ \int_{Y \setminus \{y_L(S, x) \cup y_H(S, x)\}} (p_2 S + p_1 x + p_2 B(S, x, y)) dF_{\hat{Y}}(y) \\ &+ \int_{y_H(S, x)} ((p_2 - p_C)S + (p_1 - p_C)x + (p_2 - p_C)B(S, x, y) + p_C N) dF_{\hat{Y}}(y) \end{aligned}$$

where the limits of the integration follows from the definitions of y_L and y_H . Differentiating $r(\theta, S, x)$ in x we obtain

$$\frac{\partial r}{\partial x} = \int_{Y \setminus \{y_L \cup y_H\}} \left(p_1 + p_2 \frac{\partial B(S, x, y)}{\partial x} \right) dF_{\hat{Y}}(y) + \int_{y_H} \left((p_1 - p_C) + (p_2 - p_C) \frac{\partial B(S, x, y)}{\partial x} \right) dF_{\hat{Y}}(y).$$

Setting $\partial r/\partial x = 0$ and rearranging the terms gives (5) ■

Proof of Lemma 5.

For joint concavity, observe that since r is concave in θ and in x by lemmas 3 and 4, respectively, it is sufficient to show that the determinant of the Hessian $\partial^2 r / \partial x^2 \partial^2 r / \partial \theta^2 - (\partial^2 r / \partial x \partial \theta)^2 > 0$.

For supermodularity, observe that by Theorem 7 if $g(S, x, \hat{y})$ is supermodular in (S, x, \hat{y}) , and because \hat{Y} is stochastically supermodular in (θ, S) , then $r(\theta, S, x)$ is also supermodular in (θ, S, x) . (See Section 3.10.1 in Topkis (1998)).

Consider two cases:

Case 1: if $D(\hat{y}) > N$ then $x \leq N - S \leq D(\hat{y}) - S$. From (1) we obtain

$$g(S, x, \hat{y}) = \begin{cases} p_2 S + p_1 x + p_2 B(S, x, \hat{y}), & \text{if } B - (N - S - x) < 0; \\ (p_2 - p_C)S + (p_1 - p_C)x + (p_2 - p_C)B + p_C N, & \text{if } B - (N - S - x) \geq 0. \end{cases} \quad (16)$$

If $p_C = 0$ then by the definition of B and the conditions of the Lemma, $B = Q^2(\hat{y}) - S - cx$ for some constant $c > 0$. Thus $g = p_2 Q^2(\hat{y}) + x(p_1 - cp_2)$, which is clearly supermodular. If $p_C > 0$ then by the conditions of the lemma $B = Q^2(\hat{y}) - S - x$, and thus $B - (N - S - x) < 0$ iff $Q^2(\hat{y}) < N$. Since by definition $Q^2(\hat{y}) < N$, it therefore follows that $g = p_2 Q^2(\hat{y}) + (p_1 - p_2)x$ which is clearly supermodular. In either case the derivative of g in x is not a function of \hat{y} , and so $\frac{\partial^2 r}{\partial x \partial \theta} = 0$, i.e., r is jointly concave.

Case 2: if $D(\hat{y}) \leq N$ then

$$g(S, x, \hat{y}) = \begin{cases} p_2 S + p_1 x + p_2 B(S, x, \hat{y}), & \text{if } 0 \leq x \leq D(\hat{y}) - S; \\ p_2 S + p_1 (D(\hat{y}) - S), & \text{if } D(\hat{y}) - S < x \leq N - S. \end{cases}$$

By the conditions of the lemma $B = Q^2(\hat{y}) - S - \frac{p_1}{p_2}x$. Thus $g = p_2 Q^2(\hat{y})$ for $0 \leq x \leq D(\hat{y}) - S$, and $g = p_2 S + p_1 (D(\hat{y}) - S)$ for $D(\hat{y}) - S \leq x \leq N - S$. Further, by condition (B4), $B = 0$ at $x = D(\hat{y}) - S$, i.e., $p_2 Q^2(\hat{y}) = p_2 S + p_1 (D(\hat{y}) - S)$. This implies that the values of g are equal on both intervals and so g is independent of x and therefore supermodular and jointly concave. ■

Proof of Theorem 3. In the case of excess capacity if $x > \hat{x} = A$, then $g(\alpha, y, x) = p_2(1 - \alpha)y d_2 + p_1 A$ which is independent of x .

In the case of scarce capacity by definition $A > N - S \geq x$. If $x > \hat{x}$, then $g(\alpha, y, x) = p_2 S + p_1 x + (p_2 - p_C)\alpha y d_2 \left(1 - \frac{x}{A}\right) + p_C(N - S - x)$, and so $\frac{\partial g}{\partial x} = (p_1 - p_C) - (p_2 - p_C)\frac{\alpha y d_2}{A} \leq$

0 as $p_C \geq p_1$. Therefore, $x^* \in [0, \hat{x}(\alpha, y)]$. On this interval $g(\alpha, y, x) = p_2 S + p_1 x + p_2 \alpha y d_2 (1 - \frac{x}{A})$, which is linear in x , and so the optimal solution is on the boundary of the interval; that is $x^* \in \{0; \hat{x}(\alpha, y)\} = \Pi_t$.

We next show the existence and uniqueness of the threshold waiting fraction. Let $C(\alpha) = \frac{\partial g}{\partial x}$ and observe $C(\alpha) = p_1 - \frac{\alpha Y p_2 d_2}{(a+bY)d_1 + \alpha Y d_2}$. Differentiating it we obtain $\frac{\partial C}{\partial \alpha} = -\frac{p_2 d_2 d_1 Y (a+bY)}{((a+bY)d_1 + \alpha Y d_2)^2} \leq 0$. Therefore α^* solves $C(\alpha^*) = 0$. Since $C(0) = p_1 > 0$ and C is monotonically decreasing, α^* is unique.

Finally, for $\alpha \geq \alpha^*$, $C(\alpha) \leq 0$; that is $g(\alpha, x, y)$ is nonincreasing in x and so $x^* = 0$. Otherwise the maximal revenue is attained at $x^* = \hat{x}(\alpha, y)$. In the case of excess capacity, however, the revenue function is flat on $[\hat{x}(\alpha, y), N - S]$, and so in this case any $x \in [\hat{x}(\alpha, y), N - S]$ is optimal if $\alpha \leq \alpha^*$. ■

Proof of Theorem 4. Recall that we consider two simplified models: the one with a deterministic waiting fraction, given by (13), and the one with deterministic demand, given by (14). We first prove the theorem for the deterministic waiting case, and then extend it to deterministic demand case. We require the following three lemmas.

Lemma 6 $g(\alpha, y, x)$ is decreasing convex in α for $x \in \Pi_t$.

Proof If $x = 0$ then $g(\alpha, y, 0) = p_2 y (1 - \alpha) d_2 + p_2 \alpha y d_2 = p_2 y d_2$, which is not a function of α , and therefore is decreasing convex in a weak sense.

If $x = \hat{x}$ then in the excess capacity case $g(\alpha, y, \hat{x}(\alpha, y)) = p_2 (1 - \alpha) y d_2 + p_1 ((a + by) d_1 + \alpha y d_2)$ which is linear decreasing in α since $\frac{\partial g}{\partial \alpha} = -y d_2 (p_2 - p_1) \leq 0$ because $p_2 \geq p_1$. In the scarce capacity case observe that $\frac{\hat{x}}{A} = \frac{N - S - \alpha y d_2}{(a + by) d_1 + \alpha y d_2 - \alpha y d_2} = \frac{N - S - \alpha y d_2}{(a + by) d_1}$, which is linear in α . Therefore $g(\alpha, y, \hat{x}(\alpha, y)) = p_2 y (1 - \alpha) d_2 + p_1 x + (p_2 - p_C) \frac{N - S - \alpha y d_2}{(a + by) d_1} + p_C (N - S - x)$, which is linear decreasing in α since $\frac{\partial g}{\partial \alpha} = -\frac{y d_2 (p_2 - p_1) (N - y d_2)}{(a + by) d_1} \leq 0$ as $y d_2 \leq \bar{y} d_2 \leq N$ and $p_2 \geq p_1$ by the assumption, and $\frac{\partial^2 g}{\partial \alpha^2} = 0$. ■

Lemma 7 If $f(x)$ is decreasing convex and $g(x)$ is increasing concave, then $f(g(x))$ is decreasing convex.

Proof Follows by the chain rule.

Convex: $\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial g^2} \left(\frac{\partial g}{\partial x}\right)^2 + \frac{\partial f}{\partial g} \frac{\partial^2 g}{\partial x^2} \geq 0$, because f is decreasing convex and g is concave.

Decreasing: $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial g} \frac{\partial g}{\partial x} \leq 0$, because f is decreasing, and g is increasing. ■

Lemma 8 *Let $f(x, y)$ be decreasing and(or) convex in x for all $y \in Y$, then*

(a) $\sum_{y \in Y} f(x, y)$ *is decreasing and(or) convex in x ;*

(b) $\sup_{y \in Y} f(x, y)$ *is decreasing and(or) convex in x ;*

Proof Follows from Theorems 5.2 and 5.5 in Rockafellar (1997). See also Heyman and Sobel (1984), p. 525 and Theorem A.4 in Porteus (2002), p. 226. ■

(i) Deterministic waiting model. For period T the claim is implied by Theorem 3. For $x_T \in \Pi_T$, by Lemma 6, the single-period revenue, $g_T(\alpha_T, y_T, x_T)$, is decreasing and convex in α_T . Therefore since $R_T^*(\theta_T, \alpha_T, y_T) = \max_{x_T \in \Pi_T} g_T(\alpha_T, y_T, x_T)$, R_T^* is also decreasing and convex in α_T by part (b) of Lemma 8.

In the deterministic waiting model $\alpha_T \equiv \theta_T = h_{T-1}(\theta_{T-1}, x_{T-1})$, and so by the above R_T^* is also decreasing and convex in $h_{T-1}(\cdot)$. Therefore its expectation over y_T , ϕ_T^{DW} as per (13), is decreasing and convex in $h_{T-1}(\cdot)$ by part (a) of Lemma 8.

Let $1 \leq t < T$ and suppose that for every $n \in [t+1, T]$, $\phi_n(h_{n-1}(\theta_{n-1}, x_{n-1}))$ is decreasing convex in h_{n-1} . Then by Lemma 7, in period $t+1$, ϕ_{t+1} is decreasing and convex in θ_t and x_t .

Recall that $g_t(\alpha_t, y_t, x_t)$ is piecewise linear in x_t on $[0, N - S_t]$ with the breakpoint at $x_t = \hat{x}_t$, and further recall that $g_t(\alpha_t, y_t, x_t)$ is decreasing in x_t on $x_t \in (\hat{x}_t, N - S_t]$. Since ϕ_{t+1} is decreasing and convex in x_t , $R_t(\theta_t, \alpha_t, y_t, x_t)$ consists of two adjacent segments both convex in x_t , and $R_t(\theta_t, \alpha_t, y_t, x_t)$ decreases in x_t on $x_t \in (\hat{x}_t, N - S_t]$. Thus $x_t^* \in [0, \hat{x}_t]$. Finally, since the revenue-to-go function is convex on this interval, $x^* \in \{0, \hat{x}_t\} \equiv \Pi_t$.

Therefore it is sufficient to prove that the induction assumption holds for period t ; that is that ϕ_t is decreasing and convex in h_{t-1} .

For $x_t \in \Pi_t$, the single-period revenue, $g_t(\alpha_t, y_t, x_t)$, is decreasing and convex in α_t by Lemma 6. The future revenue, ϕ_{t+1} , is also decreasing and convex in α_t by the induction assumption, upon noting that in the deterministic waiting model $\alpha_t \equiv \theta_t$. Therefore $R_t(\theta_t, \alpha_t, y_t, x_t)$ is decreasing and convex in α_t . And so by part (b) of Lemma 8, R_t^* is also decreasing and convex in α_t . Finally, since in deterministic waiting model y_t is independent of α_t , by part (a) of Lemma 8 $\phi_t = E_{y_t} [R_t^*]$ is decreasing and convex in α_t , and therefore in h_{t-1} (since $\alpha_t \equiv \theta_t = h_{t-1}$).

(ii) Deterministic demand model. For period T the claim is implied by the single-period result, given in Theorem 3.

In the deterministic demand model, observe that $R_T^*(\theta_T, \alpha_T, 1) = \max_{x_T \in \Pi_T} g(\alpha_T, 1, x_T)$ is independent of θ_T . Thus, from (14), $\phi_T^{DD} = E_{\alpha_T | h_{T-1}}[R_T^*(\alpha_T)]$, which is decreasing and convex in h_{T-1} by Theorem 7, since by the above R_T^* is decreasing in α_T , and α_T is stochastically increasing and concave in θ_T by our assumption. Therefore by Lemma 7, ϕ_T is decreasing and convex in θ_{T-1} and x_{T-1} , since $h_{T-1}(\theta_{T-1}, x_{T-1})$ is increasing and concave.

Let $1 \leq t < T$ and suppose that for every $n \in [t+1, T]$, $\phi_n(h_{n-1}(\theta_{n-1}, x_{n-1}))$ is decreasing convex in h_{n-1} . Then by Lemma 7, in period $t+1$, ϕ_{t+1} is decreasing and convex in θ_t and x_t . With this $x_t^* \in \{0; \hat{x}_t\} \equiv \Pi_t$ by the same argument as in the proof of Theorem 4, and it remains to prove that ϕ_t is decreasing and convex in h_{t-1} .

Since α_t is stochastically increasing and concave in $\theta_t = h_{t-1}(\cdot)$, the vector (θ_t, α_t) is stochastically increasing and concave in h_{t-1} .

For $x_t \in \Pi_t$, single-period revenue $g_t(\alpha_t, 1, x_t)$, is decreasing in α_t by Lemma 6 and independent of θ_t . Future revenue, ϕ_{t+1} , is independent of α_t , and is decreasing in h_t by the induction assumption, and therefore by Lemma 7 ϕ_{t+1} is also decreasing in θ_t . Thus $R_t(\theta_t, \alpha_t, y_t, x_t)$ is decreasing in (θ_t, α_t) . And by part (b) of Lemma 8, R_t^* is also decreasing in (θ_t, α_t) . From (14) $\phi_t = E_{(\theta_t, \alpha_t)}[R_t^*]$ is decreasing and convex in $h_{t-1}(\cdot)$ by Theorem 7 because (θ_t, α_t) is stochastically increasing and concave in h_{t-1} . ■

Proof of Theorem 6. Follows by the same argument as in the deterministic waiting case in the proof of Theorem 4. As in Lemma 6 we prove that $g_t(\alpha_t, y_t, x_t)$ is convex in α_t for $x_t \in \Pi_t$, where Π_t is redefined as $\{0, \hat{x}, N - S\}$.

If $x = 0$ then $g(\alpha, y, 0) = p_2 S + p_3 y \psi(\alpha) D_3$, which is convex since ψ is convex.

If $x = \hat{x}$ then in the excess capacity case $g(\alpha, y, A(\alpha, y)) = p_2(1 - \alpha)yD_2 + p_1((a + by)d_1 + \alpha y D_2)$ which is linear in α . In the scarce capacity case, if bumping is not allowed then g is piecewise linear convex. By our assumption, if $\alpha \leq \hat{\alpha}$ then $\psi(\alpha) = 0$ and so $\frac{\partial g}{\partial \alpha} = -yD_2(p_2 - p_1)$. If $\alpha > \hat{\alpha}$ then $\psi' = D_2/D_3$ and $\frac{\partial g}{\partial \alpha} = -yD_2(p_2 - p_1 E(\alpha, y)) + yD_2 p_3(1 - E(\alpha, y))$, where $E(\alpha, y) = \frac{N - S - \psi(\alpha)yD_3}{A(\alpha, y) - \psi(\alpha)yD_3} < 1$ and $\frac{\partial E}{\partial \alpha} = -\frac{y((a + by)d_1 + yD_2 - N)(D_2 - \psi'_\alpha D_3)}{(A(\alpha, y) - \psi(\alpha)yD_3)^2} = 0$. The function is convex because $\frac{\partial g}{\partial \alpha}|_{\alpha \geq \hat{\alpha}} - \frac{\partial g}{\partial \alpha}|_{\alpha < \hat{\alpha}} = yD_2(p_3 - p_1)(1 - E(\alpha, y)) \geq 0$. If bumping is allowed, then

$$\frac{\partial^2 g}{\partial \alpha^2} = \left(\frac{yD_3((a + by)d_1 + yD_2 - N)(p_3 - p_1)}{(A(\alpha, y) - y\psi(\alpha)D_3)^3} \right) \quad (17)$$

$$(A(\alpha, y)\psi''_\alpha(A(\alpha, y) - y\psi(\alpha)D_3) + 2y(\psi'_\alpha D_3 - D_2)(\psi'_\alpha A(\alpha, y) - y\psi D_2)) \geq 0$$

because the numerator in the first term in (17) is positive since in the scarce capacity case $(a + by)d_1 + yD_2 \geq N$, and $p_3 > p_1$ by the definition. The denominator is positive since

$A(\alpha, y) - y\psi(\alpha)D_3 = (a + by)d_1 + y(\alpha D_2 - \psi(\alpha)D_3) \geq (a + by)d_1 + yD_2 \geq 0$ as $\alpha D_2 \geq \psi(\alpha)D_3$ (by the definition of ψ , since the number of waiting customers of class-2 is non-negative). In the second term, $A(\alpha, y)\psi''_\alpha(A(\alpha, y) - y\psi(\alpha)D_3) \geq 0$ since ψ is convex and by the above $A(\alpha, y) - y\psi(\alpha)D_3 > 0$. And $2y(\psi'_\alpha D_3 - D_2)(\psi'_\alpha A(\alpha, y) - y\psi D_2) \geq 0$ because by the assumptions on ψ both elements of the product are negative.

If $x = N - S$ then in the case of the excess capacity $N - S > A$ and therefore $g(\alpha, y, N - S) = p_2(1 - \alpha)yD_2 + p_1((a + by)d_1 + \alpha yD_2)$, which is linear in α . In the case of scarce capacity $N - S > \hat{x}$ and therefore $g(\alpha, y, N - S) = p_2(1 - \alpha)yD_2 + p_1(N - (1 - \alpha)yD_2) + (p_3 - p_C)\psi(\alpha)yD_3(1 - (N - (1 - \alpha)yD_2)/A(\alpha, y))$. For this,

$$\frac{\partial^2 g}{\partial \alpha^2} = \left(\frac{yD_3((a + by)d_1 + yD_2 - N)(p_3 - p_C)}{A(\alpha, y)^3} \right) \quad (18)$$

$$(A(\alpha, y)^2\psi''_\alpha + 2yD_2(y\psi D_2 - \psi'_\alpha A(\alpha, y))) \geq 0$$

because the first term in (18) is positive since in the scarce capacity case $(a + by)d_1 + yD_2 \geq N$, and $p_3 > p_C$ by the definition. And in the second term, ψ is convex, and by the assumption $y\psi D_2 \geq \psi'_\alpha A(\alpha, y)$. Therefore $g(\alpha, y, x)$ is convex in α for $x \in \Pi_t$, and the result follows.

Lastly note that in the no-bumping case $p_C = p_3$ and so the above derivative is zero. Therefore g is a linear decreasing function of α , and since h increases in x , future revenue is therefore decreasing in x . Single period revenue decreases in x for $x > \hat{x}$ and therefore the total revenue also decreases on $x > \hat{x}$. Hence $x^* \in \{0, \hat{x}\}$. ■

Appendix C: Simplified Models with Non-perfectly Correlated Demands

In our simplified models we assumed that the demands between classes -1 and -2 are either independent, or perfectly positively or negatively correlated. That allowed us to present the solution in closed form. By observing initial sales, and knowing the waiting parameter, we could estimate the exact number of waiting customers.

In this section we extend the simplified models to the cases of non-perfectly correlated demands. We show that (subject to a certain regularity condition) it is sufficient to know the upper (lower) bound on the class-1 demand to be able to determine the optimal number of units on sale in the closed form in the case of positive (negative) correlation. We refer to the simplified model with two classes of Sections 5.2 and 5.3 as the *main* model.

To model non-perfectly correlated demands we assume that whenever the class-2 demand multiplier is y_t , the class-1 demand multiplier can be either the same y_t , with probability q , or ϵy_t , $\epsilon > 0$ with probability $1 - q$. Such a treatment is an extension of the high/low demand models that are frequent in the dynamic pricing literature (e.g, Su 2007). Recall that whenever the class-2 demand multiplier is y , the class-1 demand multiplier is $a + by$. For the ease of exposition in this section we assume $a = 0$ and $b = 1$; it is not difficult (but messy) to verify that the results hold for general a, b . Let $A(\alpha, y) = \alpha y d_2 + y d_1$, and let $A_\epsilon(\alpha, y) = \alpha y d_2 + \epsilon y d_1$. Redefine similarly from (10) \hat{x}_t and $\hat{x}_{\epsilon t}$. Suppose $\epsilon > 1$; observe $A(\alpha, y) < A_\epsilon(\alpha, y)$ and so $\hat{x}_t < \hat{x}_{\epsilon t}$.

From (9) denoting the expected single period revenue as $g2_t$ we obtain:

$$\begin{aligned}
 g2_t(\alpha_t, y_t, x_t) &= p_2 S_t + p_1 (q \min[x_t, A_t(\alpha_t, y_t)] + (1 - q) \min[x_t, A_{\epsilon t}(\alpha_t, y_t)]) \\
 &\quad + p_2 \alpha_t y_t D_2 \left(q \left(1 - \frac{\min[x_t, A_t(\alpha_t, y_t)]}{A_t(\alpha_t, y_t)} \right) + (1 - q) \left(1 - \frac{\min[x_t, A_{\epsilon t}(\alpha_t, y_t)]}{A_{\epsilon t}(\alpha_t, y_t)} \right) \right) \\
 &\quad - q p_C \left(\alpha_t y_t D_2 \left(1 - \frac{\min[x_t, A_t(\alpha_t, y_t)]}{A_t(\alpha_t, y_t)} \right) - (N - S_t - x_t) \right)^+ \\
 &\quad - (1 - q) p_C \left(\alpha_t y_t D_2 \left(1 - \frac{\min[x_t, A_{\epsilon t}(\alpha_t, y_t)]}{A_{\epsilon t}(\alpha_t, y_t)} \right) - (N - S_t - x_t) \right)^+ \quad (19)
 \end{aligned}$$

Note that this revenue function resembles the main model, (9), if $\epsilon = 1$ or, equivalently, $q = 1$.

In the main model we show that in each period there are only two possibly optimal solutions: $x_t = 0$ or $x_t = \hat{x}_t$. The result holds because the revenue function in the single

period is piece-wise linear with a breakpoint at \hat{x}_t , non-increasing for $x > \hat{x}_t$, and the future revenue is decreasing and convex. Thus the total revenue was a piece-wise convex function, which is decreasing for $x > \hat{x}_t$, and so the optimal solution is either $x_t = 0$ or $x_t = \hat{x}_t$.

Below we show that in this new model with non-perfectly correlated demands an equivalent result holds, but rather for $x_t = 0$ or $x_t = \hat{x}_{et}$. The logic is also similar – we show that the single-period revenue is piece-wise convex on $x \in [0, \hat{x}_{et}]$ and non-increasing for $x > \hat{x}_{et}$. That ensures that in the single period the optimal solution is either $x_t = 0$ or $x_t = \hat{x}_{et}$. Convexity of the future revenue then follows by the same argument as in the main model upon noting that \hat{x}_{et} in the new model is equal to \hat{x}_t of the old model but with nominal demand d_1 redefined as ϵd_1 .

It therefore remains to show piece-wise convexity and decreasing property for the single-period revenue function g_2 . As in the main model we proceed by considering the excess and scarce capacity cases. In the new model these capacity scenarios are transformed into three cases:

Case 1: Always excess capacity, i.e., $A_\epsilon(\alpha, y) + S \leq N$. In this case bumping does not occur and so the revenue function simplifies to:

$$g_{2t}(\alpha_t, y_t, x_t) = p_2 S_t + p_1 (q \min[x_t, A_t(\alpha_t, y_t)] + (1 - q) \min[x_t, A_{et}(\alpha_t, y_t)]) \\ + p_2 \alpha_t y_t D_2 \left(q \left(1 - \frac{\min[x_t, A_t(\alpha_t, y_t)]}{A_t(\alpha_t, y_t)} \right) + (1 - q) \left(1 - \frac{\min[x_t, A_{et}(\alpha_t, y_t)]}{A_{et}(\alpha_t, y_t)} \right) \right) \quad (20)$$

If $x \leq \hat{x}_t$ then g_2 is linear in x with slope $p_1 - p_2 \alpha y D_2 \frac{q A_\epsilon(\alpha, y) + (1 - q) A(\alpha, y)}{A(\alpha, y) A_\epsilon(\alpha, y)}$.

If $\hat{x}_t \leq x \leq \hat{x}_{et}$ then g_2 is linear in x with slope $(1 - q) \left(p_1 - \frac{p_2 \alpha y D_2}{A_\epsilon(\alpha, y)} \right)$.

If $x \geq \hat{x}_{et}$ then same as in the main model, g_2 is independent of x .

By taking the difference of the first and second slopes, it is not difficult to check that g_2 is convex on $x \in [0, \hat{x}_{et}]$ if $p_1 - \frac{p_2 \alpha D_2}{\alpha D_2 + d_1} \leq 0$. The latter condition is the familiar “bang-bang” threshold from the main model (see the proof of Theorem 3 for $a = 0, b = 1$). The inequality holds if the difference in prices, $p_2 - p_1$ is high enough so that the fraction of waiting customers α is never below a certain point that makes the threshold equal zero. If that is the case, i.e., when the discounted price is low enough, then the revenue function is convex.

Case 2: Probabilistic scarce capacity, i.e., $A(\alpha, y) + S \leq N \leq A_\epsilon(\alpha, y) + S$. In this case bumping can occur only if the class-1 demand is high (has multiplier ϵy , which happens

with probability $1 - q$). In that case the revenue function is:

$$\begin{aligned}
g2_t(\alpha_t, y_t, x_t) &= p_2 S_t + p_1 (q \min[x_t, A_t(\alpha_t, y_t)] + (1 - q) \min[x_t, A_{et}(\alpha_t, y_t)]) \\
&\quad + p_2 \alpha_t y_t D_2 \left(q \left(1 - \frac{\min[x_t, A_t(\alpha_t, y_t)]}{A_t(\alpha_t, y_t)} \right) + (1 - q) \left(1 - \frac{\min[x_t, A_{et}(\alpha_t, y_t)]}{A_{et}(\alpha_t, y_t)} \right) \right) \\
&\quad - (1 - q) p_C \left(\alpha_t y_t D_2 \left(1 - \frac{\min[x_t, A_{et}(\alpha_t, y_t)]}{A_{et}(\alpha_t, y_t)} \right) - (N - S_t - x_t) \right)^+ \quad (21)
\end{aligned}$$

For this observe that depending on the values of x , there are the same three ranges as in the Case 1 above. In the third range, $g2$ is decreasing in x same as in the main model, and in either of the first two the slope is modified by deducting the same penalty-related term (the multiple of p_C). Thus the difference between slopes is the same as in case 1, and so convexity on the $x \in [0, \hat{x}_{et}]$ range follows by the same argument.

Case 3: Always scarce capacity, i.e., $A(\alpha, y) + S \geq N$. In this case the revenue function is given by (19) and depending on the value of x the same three ranges as in Case 1 can occur. In the third range the slope in x is negative by the same argument as in the main model. The slope in the first range is same as in Case 1 with two penalty terms deducted, and the slope in the second range has only the second penalty term deducted. Thus the slope over the second range is larger; convexity over $x \in [0, \hat{x}_{et}]$ follows.

In sum, the single-period revenue function is convex on $x \in [0, \hat{x}_{et}]$ and non-increasing on $x > \hat{x}_{et}$, and so as we discussed above the results of Theorems 3 and 4 hold with $x^* \in \{0; \hat{x}_{et}\} \equiv \Pi_{et}$. The sufficient condition for this result is that $p_1 - \frac{p_2 \alpha D_2}{\alpha D_2 + d_1} \leq 0$. This inequality holds if the difference in prices, $p_2 - p_1$ is high enough so that the fraction of waiting customers α is never below a certain point that makes the threshold equal zero.

Managerially, it implies that if the discounted price is low enough, then the firm does not need to know the exact number of discount-only class-1 customers. Rather, it is sufficient to know the lower bound on the number of discount customers.

Further, it is not hard to verify that if $\epsilon < 1$ then the condition for convexity becomes a \geq , and so the logic flips – the discounted price has to be high enough so that the number of waiting customers is always low enough, and then it is sufficient for the firm to only know the upper bound on the number of discount-only class-1 customers in the market.

Appendix D: Random Allocation of Discounted Units

We tested whether it is necessary in our model to allocate inventory in a random first come, first serve (FCFS) manner or if we are justified in assuming a proportional allocation. Under FCFS, assuming all waiting customers arrive according to the same process, the allocation would be determined by the realization of a Hypergeometric random variable, implying the allocation function is given by $B_{FCFS} = \psi(\alpha)yD_3 - \text{Hypergeometric}(x, \psi(\alpha)yD_3, A)$. Under our proportional allocation assumption, $B = \psi(\alpha)yD_3(1 - \frac{\min[x,A]}{A}) = E[B_{FCFS}]$, thus proportional allocation substitutes B_{FCFS} with its expected value.

We compared the analytical optimal policy derived for proportional allocation with the policy optimal for random allocation derived using simulation. Using the scenarios presented in Section 7, we observed that the simulated and optimal policies in most cases put the same number of units on sale. When non-zero, the optimal number of units to place on sale is typically large (e.g., in Figure 2 (a) $\hat{x} = 75...90$ units). Therefore, by the central limit theorem, the relative error of using the expectation as opposed to the realization of B_{FCFS} is small. As a result, the revenue generated under the optimal policy when the discounted units are allocated in a random, FCFS manner is also close to that of the case when they are proportionally allocated. In our simulations the difference between these revenues never exceeded 3 percent. Thus we conclude that for the examples we studied, the proportional allocation assumption is justified.

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