Revenue Management with End-of-Period Discounts in the Presence of Customer Learning

Anton Ovchinnikov
Darden Graduate School of Business, University of Virginia 100 Darden Boulevard, Charlottesville VA 22903 USA
aovchinnikov@darden.virginia.edu

Joseph M. Milner
Joseph L. Rotman School of Management, University of Toronto 105 St. George Street, Toronto ON M5S 3E6 Canada
milner@rotman.utoronto.ca

Consider a firm that sells identical products over a series of selling periods (e.g., weekly all-inclusive vacations at the same resort). To stimulate demand and enhance revenue, in some periods, the firm may choose to offer a part of its available inventory at a discount. As customers learn to expect such discounts, a fraction may wait rather than purchase at a regular price. A problem the firm faces is how to incorporate this waiting and learning into its revenue management decisions.

To address this problem we summarize two types of learning behaviors and propose a general model that allows for both stochastic consumer demand and stochastic waiting. For the case with two customer classes we develop a novel solution approach to the resulting dynamic program. We then examine two simplified models, where either the demand or the waiting behavior are deterministic, and present the solution in a closed form. We extend the model to incorporate three customer classes and discuss the effects of overselling the capacity and bumping customers. Through numerical simulations we study the value of offering end-of-period deals optimally and analyze how this value changes under different consumer behavior and demand scenarios.

1. Introduction

The rapid growth of online purchases of airline tickets and other travel-related products has presented the travel and leisure industry with a number of challenges and opportunities. These include the need to develop the capability to rapidly change prices and availability of inventory, track and respond to competitor moves, and address changes in consumer behavior. This growth has also provided the capability to offer inventory that is not selling at the expected rates, so called “distressed” inventory, at a discount in the days prior to a departure of a flight or other product, such as a hotel room night, vacation package, or a weekend car rental. Such discounts while frequently referred to as last minute deals, often are offered days or weeks prior to the departure; for example, the lowest airfares are observed 3-8 weeks prior to departure (Stinger 2002).

Such end-of-period period discounts present an opportunity to purchase products at noticeably lower prices and consumers are taking advantage of the offers. For example, a quick Web search for
a week-long all-inclusive vacation for an approaching weekend in the Fall of 2006 produced choices for less than $400; this compares with prices in excess of $1,000 offered for the vacation months in advance. Increasingly, many travelers are learning to expect such end-of-period discounts and “prefer to book later in the hope of getting a good deal” (Fenton and Griffin 2004). According to American Express, “nearly half of all travelers say they intend to wait until the last minute to plan their vacations” (De Lisser 2002). Similarly, in private conversations, executives of a leading vacation tour operator noted that as a result of customer waiting for deep discounts, early bookings are “slow” and 27% of the bookings are made in the last 15 days. That is, travel firms are observing that as customers increasingly expect end-of-period discounts, additional inventory is distressed. As a result firms sell more units at a discount and lose revenue. This suggests that firms should carefully consider consumer response and incorporate it into their management policies.

The goal of this paper is to develop a stylized model that incorporates consumer response to revenue management. In this paper, we assume a travel firm (airline, car rental firm, hotel, etc.) needs to determine the number of units (seats, cars, rooms, etc.) to place on sale at at the end of each of number of a selling seasons (flights, weeks, etc.) that we refer to as periods. (Note, that the term period refers to different flights, not the slices of time during the sales of a single flight.) The objective of the firm is to maximize the total discounted revenue over the horizon. In each period, a fraction of the customers purchase at a regular, non-discounted price and a fraction waits for a potential end-of-period sale. Customers who wait, but do not receive inventory at the discounted price, may be offered inventory for purchase at a higher price. The decision made by the firm is to determine the number of units (if any) to put on sale in each period, understanding that the fraction of customers that wait for a sale in the following period is affected by the decision in the current period.

We assume the discounted price is fixed, and the firm determines the number of units available at this price. Such a quantity-oriented approach is dominant in the travel industry as opposed to a price-changing approach which is more common in other industries such as retailing (Talluri and van Ryzin 2004). We consider model variations with two and three prices in order to derive insights about the optimal polices. In practice, one would implement revenue management with more price levels; typically eight or more. Our initial model with two prices (and two customer classes) models such industries as packaged tours and performance events, where a common practice of pre-publishing prices in catalogs effectively reduces the firms’ ability to increase prices in the case of high demand. Our subsequent model with three classes captures the examples of airlines,
car rental firms, and hotels that can increase their price if there are customers willing to spend more for the product.

The fundamental contribution of this paper to the literature is the proposal and solution of a model of revenue management that incorporates consumer response to a firm’s policy. Our model is distinguished from previous work in a number of directions. First and foremost, we model a series of repetitive revenue management decisions that influence customers’ behavior for the future periods as consumers learn in this multiple period environment. The vast majority of the literature considers a single selling period (flight, etc.), and effectively ignores the possible effects on future periods. Second, we incorporate the double uncertainty of stochastic demand and stochastic consumer behavior. We show that the resulting dynamic programming model is not amenable to standard solution methodologies and make a theoretical contribution by presenting a solution methodology to a subclass of dynamic programs in which the state of the system evolves non-monotonically. Third, we derive the optimal, closed form policies for several simplified models, and through numerical studies we document the degree to which the optimal solution provides benefits over reasonable rule-based heuristics. Finally, we discuss the effects of different patterns of behavior with respect to the types and speed of learning, and with respect to the allowance of customer bumping.

The paper generates several managerial insights. First, we show that in the long-run a firm benefits by offering end-of-period discounts, even when consumers learn and react to the firm’s decisions by waiting. The optimal policy, however, greatly depends on the way customers learn. If customers self-regulate their waiting behavior (e.g., recognizing that if all customers wait, no discounts will be provided), then the firm’s policy is a passive one, where the firm puts some inventory on sale in every period. However, if the fraction of customers waiting for a discount evolves by interpolating or smoothing between its current state and some measure of the number of units placed on sale, then the firm takes an active policy. We show that a “bang-bang” policy, where the firm intermixes periods with many units on sales with those with none, is optimal. Second, we show that allowing some inventory to perish (and not be sold) may be more profitable than selling it at a discount. Third, we show that even in the absence of no-shows, overselling the capacity and bumping passengers may be an important factor in managing consumer behavior. Finally, we demonstrate that an intuitive heuristic that makes inventory available at a discount when regular sales are low performs rather poorly; in fact it does the opposite of what should be done when consumers learn and react to firms’ inventory policies.

The remainder of the paper is organized as follows. In Section 2 we position our model in the body of relevant literature. In Section 3 we introduce the model and discuss key assumptions. In Section
4 we study the optimal policy for two customer classes under the assumption of “self-regulating” learning, and present the solution to the resulting dynamic program. The case of “smoothing” learning is discussed in Section 5, where we introduce two simplifications to our general model and present their optimal policies in the closed form. In Section 6 we extend our models to the case with three customer classes, and discuss the effects of bumping on customer behavior and on the resulting optimal policy of the firm. Numerical results are presented in Section 7, followed by the conclusions and prospects for future research. Proofs are contained in the online appendix.

2. Literature Review

As revenue management has been an active area of research, we review only the literature directly related to the current study. McGill and van Ryzin (1999), Bitran and Caldentey (2003) and the recent book, Talluri and van Ryzin (2004) provide comprehensive reviews of the broader literature. Most previous research focusses on pricing and inventory policy for either a single flight or product, or a network of flights or multiple products, where strategic response by customers to the determined policy is ignored.

Relevant work that considers multiple selling seasons includes papers on intertemporal price discrimination and advance selling, such as Stokey (1979), Sobel (1984), and Conlisk et al. (1984). A summary of retail pricing can be found in Lazear (1986). Besanko and Winston (1990) and Gale and Holmes (1993) discuss the optimal price skimming by a monopolist. Dana (1999), Xie and Shugan (2001), and Tang et al. (2004) discuss advance selling. These works do not consider customer learning and in this regard are different from ours. Customer learning is often modeled through reference price effects; for example, consider Greenleaf (1995) and Popescu and Wu (2005). These models do not consider the internal dynamics of selling to several classes of customers within each selling season, which is a major feature of revenue management systems. Sen and Zhang (1999) consider a newsvendor problem with two customer classes whose price assumptions are similar to our single period model; they do not study repetitive problems and customer learning.

Recently, a number of papers (Aviv and Pazgal 2007, Cachon and Swinney 2007, Elmaghraby et al. 2004, Gallego et al. 2007, Liu and van Ryzin 2007, Zhang and Cooper 2006) consider customer behavior with respect to markdown policies. In our model the firm does not pre-commit to a price path as in these papers. We allow the firm to decide whether there should be a markdown, markup or both: a markdown could happen if all unsold inventory is put on sale, a markup could happen if no inventory is put on sale, or the price path could be a markdown followed by a markup. Further, these papers consider only a single selling period.
Several papers allow for both markups and markdowns. Asvanunt and Kachani (2007) consider the purchasing strategy of a single strategic consumer and model her waiting decision as an optimal stopping problem. They also report an empirical study demonstrating practical effectiveness of their solution. Their extension to the case with multiple strategic consumers assumes that consumers are oblivious to other strategic buyers. Levin et al. (2007) allow each consumer to consider their effect on the others, but assume that the future utility of a unit is the same for all consumers. In contrast we assume that customers’ willingness-to-pay is constant through the selling season and that the firm rations the discounted units.

Su (2007) considers a game between the firm and a fraction of consumers who act strategically and shows how the size of this fraction determines whether the firm should increase or decrease the price. Anderson and Wilson (2006) also assume that an exogenous fraction of customers will wait. These works, however, do not specify how the fraction of customers who act strategically is determined. In contrast, this is a key feature of our model, where a fraction of customers who wait is changing, as consumers learn from past decisions of the firm.

All of the above mentioned works assume a single period in which the derived equilibrium policies of the firms and customers are arrived at after a process of learning and reacting. These works do not recognize the possible dependency between the outcome of one selling season and the behavior of consumers in the future seasons, conveyed, for example, via news articles, information on websites or word-of-mouth. Our work presents a model with multiple selling periods and captures this dependency.

There are two works that consider a multiple-period setting similar to ours. First, Cooper et al. (2004) model the “spiral-down” effect and demonstrate that in a multiple-period problem with customer learning the effect of otherwise optimal (single-period) revenue management policy could be significantly diluted. Second, Gallego et al. (2007) investigate a model in which a firm’s decision about the number of units sold at a discount alters consumers’ expectations about the probability of acquiring a discounted item in the future. They demonstrate numerically that the optimal inventory policies could evolve from one period to the next in a complex fashion that cannot be captured by a single period model. Our approach is rather different from theirs; we are able to consider a broader set of cases and establish analytical results. At the same time, their results and ours demonstrate the existence of complex inter-period dynamics, suggesting that a model of the optimal multi-period pricing policy is worth studying. We turn to this next.
3. Model

Consider a sequence of identical offerings of a perishable product or service, for example, weekly all-inclusive vacations at the same resort, Wednesday morning flights from London to New York, or weekend car rentals. To distinguish between the copies of the product offerings, we say that they are offered in different periods, and there is one offering per period. In each period \( t = 1, 2, \ldots, T \), for finite \( T \), there are \( N \) units of product available. For simplicity, we treat all demands and capacities as continuous variables, and assume that for \( \Delta > 0 \), \( \Delta \) units of inventory fill \( \Delta \) units of demand.

We initially assume that there are two prices, \( p_2 \geq p_1 \), and two customer classes, such that \( p_i \), \( i = 1, 2 \), is the highest price that class \( i \) is willing to pay for the unit of product, i.e., class-1 is the lower class customer and class-2 is the higher class. (An extension to three prices/customer classes is presented in Section 6.)

Let \( Y_t = \{Y_{jt}\}_{j=1}^J \) be a vector of random variables with joint CDF \( F_Y(y) \). We assume \( Y_t \) is defined on a finite, convex set \( \mathcal{Y} \subset \mathbb{R}^J \). Let \( Q^i_t(Y_t) \), \( i = 1, 2 \) be the demand for class-\( i \) given \( Y_t \) and let total demand \( D_t(Y_t) = Q^1_t(Y_t) + Q^2_t(Y_t) \). To avoid trivial cases when all capacity is sold and hence no discounts are possible we assume \( Q^2_t(Y_t) < N \) for all \( Y_t \in \mathcal{Y} \), i.e., the demand from class-2 is less than the capacity for all demand realizations.

As in other models in the literature, some class-2 customers may wait for a discount that may be offered to attract class-1 customers at the end of the period. In line with earlier works (e.g., Pfeifer 1989) we refer to these diverting customers as “shoppers.” However, a key feature of our model is that their waiting behavior changes over time in response to the firm’s decisions. We assume that consumer waiting behavior is described by a waiting parameter, \( \theta_t \), that represents the propensity of customers to wait for a discount. For example, there could be a random fraction, \( \alpha_t \), of class-2 customers waiting and \( \theta_t \) could be the average fraction waiting (this is the construct we use in our simplified models). The waiting parameter \( \theta_t \) changes over time capturing customer learning.

We assume that \( \theta_t \) is known at the beginning of period \( t \). The dynamics of the selling period are depicted in Figure 1.

We divide class-2 demand in two groups: \( S_t \) customers who purchase at price \( p_2 \) at the start of period \( t \) and \( M_t \) shoppers who wait for a potential discount towards the end of the period, where \( S_t(Y_t, \theta_t) + M_t(Y_t) = Q^2_t(Y_t) \). The firm initially sets the price at \( p_2 \) and observes the initial sales, \( S_t \). Class-1 customers also wait for a potential discount but do not act strategically – they only purchase if price \( p_1 \) is offered. They are similar to the “bargain-hunters” in Cachon and Swinney (2007). At a point in time close to the end of period \( t \), denoted as the sale time, the firm determines
End-of-period Inventory, \( t \), \( N \)
x units available at a discount
Sale time of period \( t \)
Beginning of period \( t \)
Beginning of period \( t+1 \)
\( M_t \)
\( B_t(S_t, x_t, Y_t) \)
\( S_t \mid \theta_t \)
\( \hat{Y}_t \mid \theta_t, S_t \)
\( \theta_{t+1} = h(\theta_t, x_t) \)
\( S_{t+1} \mid \theta_{t+1} \)
\( \text{Overflow demand } [B_t - (N - S_t - x_t)] \)
With bumping some units bought at \( p_1 \) are denied and the overflow demand is accommodated.
Without bumping the overflow demand is lost.

Figure 1 Timeline of the model.

\( x_t \), the number of unsold units on put on sale at price \( p_1 \), \( x_t \geq 0 \). If \( x_t \geq Q_1^t + M_t \), then all waiting demand is satisfied. Otherwise, let \( B_t(S_t, x_t, Y_t) \) be the number of unserved class-2 shoppers.

Once all \( x_t \) discounted units are sold, the firm again offers units for sale at price \( p_2 \) to accommodate unserved class-2 customers (or in the extension, in Section 6, at higher price \( p_3 \)). In some industries, for example in airlines, it is common to oversell the capacity, forcing some customers to be “bumped”, i.e., displaced from their seat by another, typically higher paying, customer. In such cases with bumping we assume that if \( B_t \) exceeds the number of unsold units, \( (N - S_t - x_t) \), customers that purchased at a discount are bumped; their units are sold at price \( p_2 \) and the firm incurs a penalty \( p_C \) per unit bumped. In other industries, for example in cruise lines, only unsold units may be sold at this time. This may occur because of competitive norms, legislation or regulation, or for logistical reasons, such as when a physical object is sold. Letting \( p_C = p_2 \) accomplishes this goal as there is no economic justification for selling additional units. Thus in cases without bumping the potential demand, \( B_t - (N - S_t - x_t) \) is lost.

Observe that there are incentives for class-2 customers to purchase early. Without bumping class-2 customers may not be served and presumably they would prefer to purchase at \( p_2 \) rather than not. With bumping, class-2 customers that purchase a discount seat are subject to bumping. We assume that bumped customers cannot repurchase a unit in the same period – this is a reasonable assumption supported by discussions with travel industry personnel (see discussion in Section 3.1).

The total revenue of the firm net the bumping cost for period \( t \) given \((S_t, x_t, Y_t)\) is

\[
g_t(S_t, x_t, Y_t) = p_2 S_t + p_1 \min[x_t, D_t(Y_t) - S_t] + p_2 B_t(S_t, x_t, Y_t) - p_C (B_t(S_t, x_t, Y_t) - (N - S_t - x_t))^+. \tag{1}
\]

For notational convenience let \( \hat{Y}_t \equiv Y_t | (\theta_t, S_t) \) represent the conditional random vector \( Y_t \) having observed \( S_t \) and knowing \( \theta_t \). The expected single-period revenue in period \( t \) is then given by:


$$r_t(\theta_t, S_t, x_t) = \int_y g_t(S_t, x_t, y) dF_{Y_t}(y).$$

where $F_{Y_t}(y) \equiv F_{Y_t|\theta_t,S_t}(y)$ is the cdf of $Y_t$.

We refer to the 2-vector $(\theta_t, S_t)$ as to the state of the system. We assume that the system evolves based on the decision $x_t$ according to a function $h(\theta_t, x_t)$, defining $\theta_{t+1}$, and a random draw of $S_{t+1}$ from the distribution of future sales, $F_{S_{t+1}|\theta_{t+1}}(\cdot)$. We refer to $h(\cdot)$ as the learning function because it reflects the changes in the waiting behavior of the customers as they learn about the policy of the firm. We define two types of the learning functions:

(i) smoothing, if $\frac{\partial h}{\partial x} \geq 0$ and $\frac{\partial h}{\partial \theta} \geq 0$, e.g., $h(\theta, x) = \lambda \frac{x}{N} + (1 - \lambda)\theta$ for $0 \leq \lambda \leq 1$;

(ii) self-regulating, if $\frac{\partial h}{\partial x} \geq 0$ and $\frac{\partial h}{\partial \theta} \leq 0$, e.g., $h(\theta, x) = \kappa + \lambda \frac{x}{N} - (1 - \lambda)\theta$ for $\{\kappa, \lambda > 0 | \kappa + \lambda \leq 1\}$.

The objective of the firm is to maximize the expected $T$-period revenue, discounted at a fixed rate $\delta \in (0, 1)$. Therefore, the firm determines the number of units on sale, $x_t$, for each period $t = 1, 2, ..., T$ and some initial $\theta_1$, by solving the following dynamic program

$$J_t(\theta_t, S_t, x_t) = r_t(\theta_t, S_t, x_t) + \delta E_{S_{t+1}|\theta_{t+1}}[J^*_{t+1}(\theta_{t+1}, S_{t+1})]$$

where

$$J^*_t(\theta_t, S_t) = \max_{0 \leq x_t \leq N-S_t} J_t(\theta_t, S_t, x_t)$$

subject to

$$J^*_T(\theta_{T+1}, S_{T+1}) = 0 \quad \text{for all } (\theta_{T+1}, S_{T+1})$$

$$\theta_{t+1} = h_t(\theta_t, x_t)$$

We refer to (3) as to the general model, as it expresses the uncertainty in the overall demand as well as in the fraction of class-2 customers waiting in every period. In Sections 5.1 and 6 we consider simplified models, where alternately one or the other of these uncertainties is removed.

We approach the problem by establishing properties of $J_t(\theta_t, S_t, x_t)$, in particular concavity and supermodularity. We rely on Theorem 7 in the Appendix that establishes property-inducing stochastic transformations. To do so, we make several assumptions on the demand and stochastic ordering of the random variables. See Topkis (1998), sections 2.2 and 3.9.1, for appropriate definitions of increasing functions and stochastic ordering for distribution functions in $\mathbb{R}^n$. We assume that demand function $D_t(Y_t)$ is increasing in $Y_t$. We assume that (i) $S_t$ is stochastically decreasing in $\theta_t$; (ii) $\hat{Y}_t$ is stochastically increasing in $S_t$; and (iii) $\hat{Y}_t$ is stochastically increasing in $\theta_t$.

These assumptions are consistent with the following intuitive observations. Since $\theta_t$ measures the propensity of customers to wait, the number of customers who purchase (i.e., do not wait), $S_t$, should decrease in $\theta_t$. Similarly, more customers purchasing at the initial price implies increased $Q^0_t(Y_t)$. Defining $Q^0_t$ to be increasing in at least one element of $Y_t$ and not decreasing in any, implies $\hat{Y}_t$ increasing in $S_t$. The distribution of the conditional random variable $\hat{Y}$ is dependent on $\theta$. Upon

\[ Y_t \sim \text{dist.} \]
observation of the same sales, $S_t$, and increase in $\theta$ would imply greater class-2 demand leading to a (stochastic) increase in $\hat{Y}$.

### 3.1. Discussion

**Aggregate Demand.** We consider a model where rather than tracking the detailed arrival dynamics of individual customers, the firm focuses on the aggregate behavior of customer classes through the parameter $\theta$. Our approach is motivated by many discussions with revenue management executives who noted that in a multiple-period setting like ours, customers who purchase products in different periods are typically different individuals, and it is rather unclear how these individuals react to the firm’s revenue management policy or even if they have accurate information about it at all. At the same time, the firm is concerned with the aggregate outcome of these individual behaviors, and as these executives noted, there exists information sources such as industry reports, news articles, and websites, through which aggregate demand reacts to the revenue management policy of the firm. Our model expresses such considerations.

**Waiting Parameter and Waiting Fraction.** Our key differentiating assumption from previous work is that the fraction of class-2 customers who wait is governed by a parameter, $\theta_t$, that changes with customer learning. Among the factors that lead to an environment where some customers wait and some buy early are anxiety/risk and anticipation. Customers who wait for an end-of-period deal may experience anxiety and risk because, as noted above, they are not guaranteed a product. Researchers in marketing (e.g., Nowlis et al. 2004) and economics (e.g., Loewenstein 1987) showed that for “pleasure” products, of which a vacation is an archetypal example, there exists a positive utility of anticipation. For a constant price, customers who purchase a product earlier obtain a higher utility from consuming it. We note that the utility of anticipation, widely recognized by executives in the vacation industry as one of the drivers of early purchases, creates another tradeoff against waiting. Heterogeneity of the customer population with respect to valuing such time and risk tradeoffs (Chesson and Viscusi 2000) naturally yields the aggregate outcome that some customers wait and some purchase early, which is observed in practice and is reflected in our model through waiting parameter $\theta_t$.

**Bumping.** While bumping of passengers is typically attributed to an overestimation of the number of travelers who do not show up for a flight, there is full incentive for a travel firm to sell seats for higher revenue at the last minute without considering no-shows, e.g., Bell 2008. Though travel firms often first seek for volunteers to be bumped (in exchange for future travel vouchers), involuntary bumping has become more common lately (Bailey 2007). Such customers have little
recourse other than accepting the alternate arrangements proffered by the firm plus any accompanying compensation (as of the summer of 2007 the regulated maximum compensation for U.S. domestic flights was $400.) We assume a class-2 customer denied a unit cannot re-book by paying the difference \( p_2 - p_1 \); this assumption reflects the current practice in travel industry\(^1\).

**Allocation of Discounted Units.** In practice, discounted units are sold on a “first come, first serve” basis and the class of the customer is not known. Thus, \( B_t(S_t, x_t, Y_t) \) is a result of a random draw in which, for example, all waiting customers could have equal probability of buying a discounted product; then \( B_t(\cdot) \) would have a hypergeometric distribution. However, incorporating random allocation leads to an untractable model, in part, because it requires treating the demands and capacities as integers. Therefore, we consider deterministic allocation mechanisms. In Sections 4 and 5, respectively, we discuss two forms of proportional allocation depending on the nominal or realized demands (Talluri and van Ryzin 2004, pp. 330 call such mechanisms “proportional rationing”). Our proportional allocation based on realized demands simplifies \( B_t(\cdot) \) to be equal to the expected outcome of the above-mentioned random allocation. In Appendix C, through numerical simulations, we document that the optimal policy obtained with such a simplification is very robust: that is, the revenue generated by such a policy is only marginally different from that resulting from the policy optimal under random allocation.

**Fixed Discounted Price \( p_1 \).** In general, firms could determine both the quantity to discount and the sale price for each period. Analyzing both variables simultaneously, however, leads to a very complex multi-period model, since doing so requires understanding the customer’s perception of how quantity complements/substitutes for price. For example, we would need to make assumptions on the effect on future customers for the case when the firm offered only few units on sale, but the price was very low, as opposed to the case when the firm put many units on sale, but the discount was small. Therefore, in attempt to create a stylized parsimonious model we assume that the \( p_i \)’s are fixed for the entire \( T \) periods and concentrate on the quantity decision. Furthermore, research has shown that a heuristic that charges the properly chosen single price instead of a dynamic

\(^1\) We asked managers who deal with bumping in major travel firms to comment on the realism of the non-rebooking assumption. Here are two responses we received:  
Respondent 1 “A person in the economy cabin [who bought the lower fare] would not be able to pay more money to be the person not denied boarding in an oversold situation. ... [S/he] will not have the opportunity to ‘jump ahead’ of the other passengers by paying more or upgrading.”  
Respondent 2 “In this age of fewer employees working harder than ever before, gate agents are generally solely focused on the operation—that is, getting the flight out on time. That means they will almost always decline to do any ticket transactions at the gate. ... 90-95% of the time, the ‘rich’ customer who asks the agent to upgrade his ticket to get a higher [boarding] priority will likely be told no.”

A third respondent verbally confirmed these statements. All three managers also confirmed that the consumers who paid lowest fares are the first candidates for denying boarding.
price often performs just marginally suboptimally (e.g., Gallego and van Ryzin 1994). In Section 7.2 we demonstrate how one might determine the optimal static discount price while dynamically optimizing the quantity to discount.

Learning Behaviors. Finally, we investigate both self-regulating and smoothing \( h(\theta_t, x_t) \) functions, in Sections 4 and 5, respectively. For “smoothing” functions the next period’s waiting parameter, \( \theta_{t+1} \), lies between \( \theta_t \) and \( x_t/N \), so that the decision \( x_t \) is “smoothed” into the previous belief, \( \theta_t \). Smoothing functions represent the standard moving average forecasting and are frequently used (e.g., Greenleaf 1995, Popescu and Wu 2005). Alternately, “self-regulating” functions reflect the following behavior: as the total number of waiting customers increases, the chances to obtain a product on sale decrease for an individual customer. This negatively affects the number of customers waiting – an expression of self-regulation.

In Section 4 we show that for the case with a self-regulating learning function, the revenue function is concave in the number of units on sale, \( x_t \), for every \( t \). To do so we develop the necessary methodology to prove concavity and show that in addition to being concave, the expected single-period revenue \( r_t(\theta_t, S_t, x_t) \) is also required to be supermodular and increasing. We note that our approach to showing concavity differs from standard methodologies, such as Topkis (1998) Section 3.9.2, Putterman (1994) Section 4.7.3, or their recent extensions, e.g., Smith and McCardle (2002).

In their models the state of the system evolves monotonically in the previous state and decision. That is, the components of the state vector either increase or decrease in the previous state and decision. In our general model (3), state transitions are not monotonic, since \( S_t \) is stochastically decreasing in \( \theta_t \), while \( \theta_t \) is increasing in either \( \theta_{t-1} \) or \( x_{t-1} \), or both.

In Section 5 we study the case with a smoothing learning function and show that in general concavity does not hold, unless the speed of customer learning is “slow.” Then we consider two simplifications to the general model and derive their solutions in closed form. Section 6 presents a model with three customer classes.

4. Optimal Policy for Self-Regulating Learning

In this section we derive the conditions under which the revenue-to-go function is concave when the learning function, \( h(\cdot) \), is self-regulating. Concavity implies that the optimal policy for the firm places some units on sale in each period, relying on the consumer behavior regulating the number of customers waiting.

We organize this section as follows. First in Section 4.1 we show that \( J_t(\theta_t, S_t, x_t) \) is concave, supermodular and increasing under that assumption that these properties hold for the single-period expected revenue function, \( r(\theta, S, x) \). Then in Section 4.2 we discuss the properties of the
revenue function \( g(S, x, y) \), given in (1) that ensure concavity, supermodularity and monotonicity of \( r(\theta, S, x) \). We conclude by presenting an example.

### 4.1. Concavity in Dynamic Programs With Nonmonotonic State Transitions

Observe from (3a) that since \( \theta_{t+1} = h_t(\theta_t, x_t) \) is independent of \( S_t \), the expected future revenue, \( E_{S_{t+1}|\theta_{t+1}} [J_{t+1}^* (\theta_{t+1}, S_{t+1})] \), does not depend on \( S_t \) and therefore, letting \( \phi_{t+1}(h_t(\theta_t, x_t)) = E_{S_{t+1}|\theta_{t+1}} [J_{t+1}^* (\theta_{t+1}, S_{t+1})] \) we can substitute

\[
J_t(\theta_t, S_t, x_t) = r_t(\theta_t, S_t, x_t) + \delta \phi_{t+1}(h_t(\theta_t, x_t))
\]

where the function \( \phi_{t+1} \) can be interpreted as the expected future revenue.

We make the following four assumptions which we show hold in the next section. We assume that \( r_t(\theta_t, S_t, x_t) \) is (A1) jointly concave in \( (\theta_t, x_t) \), (A2) supermodular in \( (\theta_t, S_t, x_t) \), (A3) increasing in \( S_t \), and (A4) \( S_t \) is stochastically decreasing and concave in \( \theta_t \). We also make the fundamental assumption of Section 4 (A5) \( h_t(\theta_t, x_t) \) is linear self-regulating, i.e., \( \frac{\partial h}{\partial x} \frac{\partial h}{\partial \theta} \leq 0 \), \( \frac{\partial^2 h}{\partial x^2} = \frac{\partial^2 h}{\partial \theta^2} = \frac{\partial^2 h}{\partial x \partial \theta} = 0 \).

Concavity and supermodularity in (4) are related by the following Lemma (all proofs are presented in the appendix):

**Lemma 1** If \( \phi \) is concave in \( h \), then \( J \) is concave in \( x \) and supermodular in \( (\theta, S, x) \).

Therefore it is sufficient to show that \( \phi \) is concave in \( h \), which in our original notation corresponds to \( E_{S_t|\theta_t} [J_t^* (\theta_t, S_t)] \) being concave in \( \theta_t \). Let \( f_{S_t}(y|\theta_t) \) be the density of \( S_t \) given \( \theta_t \). Then \( E_{S_t|\theta_t} [J_t^* (\theta_t, S_t)] = \int J_t^* (\theta_t, y) f_{S_t}(y|\theta_t) dy \). Concavity of this integral is established by the following Lemma, which extends Theorem 3.9.1 of Topkis (see the appendix) to the case where the integrant depends on the parameter.

**Lemma 2** Let a family of univariate random variables \( X_\theta \) with cdf \( F_X(x; \theta) \) and density \( f_X(x; \theta) \) be stochastically decreasing and concave in scalar parameter \( \theta \). Let \( v(\theta, x) \) be supermodular in \( (\theta, x) \), increasing in \( x \) and concave in \( \theta \). Then \( \int v(\theta, x) dF_X(x; \theta) \) is concave in \( \theta \).

Our main result in this section is given by the following Theorem.

**Theorem 1** Under assumptions A1 – A5, \( J_t(\theta_t, S_t, x_t) \) is concave in \( x_t \) and supermodular in \( (\theta_t, S_t, x_t) \) for all \( t = 1, 2, ... T \).

Problem (3) defines a subclass of dynamic programs for which a vector of the system state is not monotonic in the previous period’s state and decision. In this section we have shown how
to establish concavity for such dynamic programs. A similar logic with a different set of initial assumptions could be used to prove other properties, for example, convexity. We also note that to our knowledge there is no published research dealing with concavity (convexity) in the dynamic programs with nonmonotonic transitions. As such this is a technical contribution of our paper.

Next we discuss the underlying conditions on the customer behavior which ensure concavity.

4.2. Sufficient Conditions for Concavity of $J_t(\theta_t, S_t, x_t)$

In this section we study the properties of $B(S, x, \hat{Y})$ and the other parameters of the model that ensure that the expected single-period revenue function $r(\theta, S, x)$ is concave, supermodular, and increasing, so that by Theorem 1 the revenue-to-go, $J_t(\theta_t, S_t, x_t)$, is concave for every period. Since the discussion relates to a single period, time indices are omitted.

$B(S, x, \hat{Y})$ determines the number of class-2 “shoppers” who remain waiting once all discounted units are sold. That is, $B$ is the initial number of class-2 shoppers, $Q^2(\hat{y}) - S$, minus those who bought at a discount, where the latter reflects allocation of discounted units. Therefore $B$ should satisfy the following intuitive conditions:

**B1:** $B$ is increasing in $\hat{Y}$ and decreasing in $S$ and $x$; **B2:** $\partial B/\partial x \geq -1$; **B3:** $\partial B/\partial S \geq -1$; **B4:** if $x \geq D(\hat{y}) - S$ then $B(S, x, y) = 0$; and **B5:** $B(S, x, y)$ is piecewise concave in $x$ on $[0, D(\hat{y}) - S)$ and $[D(\hat{y}) - S, N]$.

The increasing/decreasing properties of (B1) follow directly from the definition. Conditions (B2) and (B3) hold because the discounted units are allocated between class-1 and -2 customers: if additional $\Delta$ discounted units are available, some of them will be sold to class-1 customers, and so the number of class-2 customers remaining can decrease by at most $\Delta$, and similarly, if additional $\Delta$ class-2 customers purchased early, then the chances of waiting class-2 customers to obtain discounted units decrease (because the number of class-1 customers does not change), thus the number of unserved class-2 customers decreases by at most $\Delta$. (B4) implies if sufficient capacity is available to accommodate all customers, then no customers remain. Finally, (B5) implies the rate of decrease in the number of unsatisfied customers (weakly) increases with the number of discounted units, e.g., as would be the case if units are allocated randomly.

Consistent with the intuitive observations of consumer behavior as per the discussion in Section 3 we assume $\hat{Y}$ is stochastically increasing and concave in $S$ and in $\theta$ and that $\hat{Y}$ is stochastically supermodular in $(\theta, S)$. We have the following Lemmas:

**Lemma 3** $r(\theta, S, x)$ is increasing in $S$ and concave in $\theta$.

**Lemma 4** $r(\theta, S, x)$ is concave in $x$ if $D(\hat{y}) > N$ for all $\hat{Y}$ or if $\frac{\partial B}{\partial x} \geq -p_1/p_2$. 
Lemma 4 implies that if there is always sufficient demand to fill the capacity the revenue function is concave. Otherwise, the assumption $\partial B/\partial x \geq -p_1/p_2$ implies that for a $\Delta$ increase in $x$, at most $p_1/p_2 \times 100\%$ is allocated to class-2 customers and $(1 - p_1/p_2) \times 100\%$ goes to class-1. For example, if discounted inventory is allocated in proportion to demand, the ratio of class-1 to class-2 customers should be greater than $(p_2 - p_1)/p_2$. If demand is low and $\partial B/\partial x < -p_1/p_2$, then for each additional unit of $x$, the additional sales at the lower price cannibalize too much of the revenue from the waiting class-2 customers. In this case, offering no discounts is optimal, though concavity in the expected revenue cannot be assured. We note that concavity of $r(\theta, S, x)$ in $x$ could hold in this case if the distribution of $\hat{Y}$ places sufficiently small probability on $Y' = \left[\hat{y} | D(\hat{y}) \leq N, \frac{\partial B}{\partial x} \leq -p_1/p_2 \right]$.

The optimal number of units on sale in the single period can be determined by solving the first-order condition $\frac{\partial r(\theta, S, x)}{\partial x} = 0$, and is given by the following corollary:

**Corollary 1** The single-period optimal number of units on the end-of-period sale, $x^*$, satisfies

$$p_1 \text{Prob}(D(\hat{Y}) > S + x^*) + p_2 \int_{y \in y_L} \frac{\partial B(S, x, y)}{\partial x} dF_{\hat{Y}}(y) = p_C \left( \text{Prob}(B(S, x^*, \hat{Y}) > N - S - x^*) + \int_{y \in y_H} \frac{\partial B(S, x, y)}{\partial x} dF_{\hat{Y}}(y) \right)$$

where $y_L$ and $y_H$ are the sets of “low” and “high” demand outcomes (see Appendix).

Recalling $\partial B/\partial x < 0$, the first-order condition implies $x$ should be chosen to balance the marginal additional revenue gained from bargain-hunters (class-1) less the marginal diverted revenue from class-2 customers who purchase at $p_1$ with the marginal penalty costs less the penalty costs saved by serving additional class-2 customers. That is, (5) expresses the newsvendor-like balance:

$$p_1 \text{Prob}((S + x^*)^{th} \text{unit is sold at } p_1) - p_2 \text{Exp}[\text{Marginal decrease in # of diverted customers}] = p_C (\text{Prob}(\text{Bumping a customer}) - \text{Exp}[\text{Marginal decrease in # of bumped customers}])$$

Joint concavity and supermodularity of $r(\theta, S, x)$ are discussed in the following Lemma:

**Lemma 5** $r(\theta, S, x)$ is supermodular in $(\theta, S, x)$ and jointly concave in $(\theta, x)$ if either (i) $D(\hat{y}) > N$ for all $\hat{y}$ in the support of $\hat{Y}$ and $\frac{\partial B}{\partial x} = \text{const}$ ($\frac{\partial B}{\partial x} = -1$ if $p_C > 0$); or (ii) $D(\hat{y}) \leq N$ for all $\hat{y}$ in the support of $\hat{Y}$, and $\frac{\partial B}{\partial x} = -p_1/p_2$.

A direct corollary to the above Lemma is that to achieve joint concavity and supermodularity the model must be restricted to the cases of only “high” demand ($D(\hat{y}) > N$) or only “low” demand ($D(\hat{y}) \leq N$). Summarizing, we have the following theorem:
Theorem 2 The revenue-to-go function is concave in $x_t$ under conditions (B1) – (B5) if either (i) total demand always exceeds capacity and the marginal discounted unit is allocated between customers classes in a constant proportion (the marginal unit is sold to a class-2 customer if the penalty cost is positive); or (ii) total demand is always less than capacity and the marginal discounted unit is allocated between the classes in proportion to their revenues.

As an example of a function that satisfies the assumptions, suppose that demands from class-1 and -2 customers are given by multiplying nominal demands, $d_1$ and $d_2$ by a random variable, $Y$. Demand at price $p_2$ is $yd_2$ and at price $p_1$ is $g(d_1 + d_2)$. Let $y$ be the minimum value of the support of $Y$. For the case with $p_C = 0$ and $g(d_1 + d_2) \geq N$, consider $B(S, x, \hat{y}) \equiv \hat{y}d_2 - S - x\frac{d_2}{d_1 + d_2}$. Here, the number of remaining class-2 customers reflects the total number of class-2 customers that wait, $M = \hat{y}d_2 - S$, net the number of class-2 customers that purchased product at the end-of-period sale. Further, the discounted units are allocated on proportion, which is constant and depends on the nominal demands, $d_2/(d_1 + d_2)$. With this, $g(S, x, \hat{y}) = \hat{y}p_2d_2 + x\left(p_1 - p_2\frac{d_2}{d_1 + d_2}\right)$, which is supermodular in $(S, x, \hat{y})$, and linear in $(x, \hat{y})$. Therefore the conditions of Theorem 2 hold and the expected revenue-to-go function is concave for every period, and so the optimal number of units on sale is easy to find.

Our results also suggest a neat interpretation of the optimal policy for the case with self-regulating learning. First, concavity implies that the firm places some units on sale in every period. Furthermore, since the revenue function is supermodular, the number of units on sale increases in the waiting parameter. But, the self-regulating learning behavior controls the number of customers waiting in the subsequent period so that it does not continue to increase. The firm takes a passive role, placing some units on sale, and relying on the consumer behavior to control future waiting. This is not the case for smoothing learning functions, where the firm must actively manage consumer waiting as we discuss below.

5. Optimal Policy for Smoothing Learning Function

In this section we assume that the learning function $h_t(\theta_t, x_t)$ is smoothing; that is, the next period’s waiting parameter, $\theta_{t+1}$, increases in both the current waiting parameter, $\theta_t$, and the number of units on sale in period $t$, $x_t$. We show that in the general model the revenue-to-go function is not necessarily concave, unless the speed of consumer learning is “slow” as defined below. To address the problems with arbitrary speed of learning we present two simplified models and show that for either simplification, the optimal policy has a “bang-bang” structure where the firm alternately places a number, $\hat{x}_t$, or zero units on sale. We describe this optimal policy in the closed form.
Under the assumption that the learning function is linear, concavity and supermodularity of the revenue-to-go require, respectively,

$$\frac{\partial^2}{\partial x^2} J(\theta, S, x) = \frac{\partial^2}{\partial x^2} r(\theta, S, x) + \delta \frac{\partial^2 \phi}{\partial h^2} \left( \frac{\partial h}{\partial x} \right)^2 \leq 0 \text{ and }$$

$$\frac{\partial^2}{\partial x \partial \theta} J(\theta, S, x) = \frac{\partial^2}{\partial x \partial \theta} r(\theta, S, x) + \delta \frac{\partial^2 \phi}{\partial h^2} \frac{\partial h}{\partial x} \frac{\partial h}{\partial \theta} \geq 0. \quad (6)$$

In the case of a self-regulating learning function, both inequalities hold if $\frac{\partial^2 \phi}{\partial h^2} \leq 0$, since by definition a self-regulating learning function satisfies $\frac{\partial h}{\partial x} \frac{\partial h}{\partial \theta} \leq 0$. In the case of a smoothing learning function, however, $\frac{\partial h}{\partial x} \frac{\partial h}{\partial \theta} \geq 0$, and so concavity and supermodularity of the revenue function place contradictory requirements on $\frac{\partial^2 \phi}{\partial h^2}$. Therefore from Theorem 1, we conclude that the revenue function is not necessarily concave.

Observe that $\partial h/\partial x$ reflects the speed at which customers learn about the firm’s decisions. Assuming $r$ is concave, from (6) if $\partial h/\partial x$ is small enough then $J$ would be concave; that is, if customer learning is slow enough, then concavity of the expected revenue function for every period are equivalent to the corresponding single-period property, and so can be established by Lemma 4.

Slow learning has been documented in the works on reference price learning with respect to the sales promotions. Greenleaf (1995) and Hardie et al. (1993) studied point-of-sales data for such commodities as peanut butter and refrigerated orange juice, and reported an analog of our $\partial h/\partial x$ to be at 0.075 and 0.17 respectively. However, we know of no research regarding the speed of learning for discounts in services. This is of interest for future research.

The speed of learning also plays an important role in Gallego et al. 2007. Their numerical simulations suggest that in the case of slow learning, the firm should offer a constant sales limit (number of units on sale), and otherwise sales limit should be raised and lowered in alternate periods. Next we prove a similar result analytically and provide a closed form expression for the number of units on sale.

### 5.1. Simplified Models

In this section we simplify the general model so that upon observing the initial sales, the firm can infer the exact number of customers waiting for period $t$. The future demand and purchasing behavior remain stochastic. This simplification allows us to solve the problem in the closed form while utilizing a less constrained $B(S, x, y)$ and relaxing the linearity assumption of the learning function.

For the remainder of the paper, let $Y \in [\underline{y}, \bar{y}]$ be a random variable and let demand from each class reflect a nominal demand, $d_i$, $i = 1, 2$ multiplied by $Y$. For class-2, demand is $d_2 Y_t$ and for
class-1, demand is \( d_1(a + bY_t) \), where \( a, b \) are given constants. Observe that with \( b > 0 \) class-1 and class-2 demands are positively correlated, with \( b < 0 \) then are negatively correlated and for \( a > 0, b = 0 \) class-1 demand is a constant. Note that in the two former cases demands are perfectly correlated. An extension to the non-perfectly correlated demands in presented in Appendix B. With these, the total demand from both classes is \( D_t(Y_t) = Y_tD_2 + (a + bY_t)d_1 \). We assume \( \theta d_2 \leq N \) and \( d_2 + bd_1 \geq 0 \) to ensure that \( D \) is increasing in \( Y \).

Let \( \alpha_t \in [a, \pi] \subseteq [0, 1] \) be the fraction of the class-2 demand that waits for the end-of-period sale in period \( t \). We refer to \( \alpha_t \) as the waiting fraction. Observe that \( M_t = \alpha_t Y_t d_2 \) and \( S_t = (1 - \alpha_t)Y_t d_2 \).

In this section we consider two simplifications:

(i) Deterministic waiting fraction model, in which the demand multiplier, \( Y_t \), is stochastic, but its distribution does not depend on the waiting parameter; the waiting fraction is deterministic with \( \alpha_t \equiv \theta_t \), and evolves according to a smoothing concave learning function \( \alpha_{t+1} = h_t(\alpha_t, x_t) \);

(ii) Deterministic demand model, in which \( Y_t \equiv \text{Constant} \) for all \( t = 1, 2, ..., T \) and w.l.o.g. we set \( Y_t \equiv 1 \); the random waiting fraction, \( \alpha_t \), is stochastically increasing and concave in the waiting parameter \( \theta_t \), which evolves according to a smoothing concave learning function \( \theta_{t+1} = h_t(\theta_t, x_t) \).

Let \( A_t(\alpha_t, Y_t) \) be the actual demand for the discounted seats. \( A_t(\alpha_t, Y_t) = D_t(Y_t) - S_t = ad_1 + Y_t(d_2 + bd_1) - S_t \). Since the firm puts \( x_t \) units on the end-of-period sale, \( \min[x_t, A_t] \) units are sold at the discounted price \( p_1 \). Assuming that the discounted inventory is allocated proportionally between class-1 and class-2 customers based on their realized demands, the number of class-2 customers that purchase discounted units is \( \min[x_t, A_t] \frac{\alpha_t Y_t d_2}{A_t} \). The number of unserved class-2 customers after the sale is

\[
B_t(\alpha_t, Y_t, x_t) = \alpha_t Y_t d_2 \left( 1 - \frac{\min[x_t, A_t]}{A_t} \right),
\]

and from (1) the net single-period revenue is

\[
g_t(\alpha_t, Y_t, x_t) = p_2 S_t + p_1 \min[x_t, A_t] + p_2 \alpha_t Y_t d_2 \left( 1 - \frac{\min[x_t, A_t]}{A_t} \right) - p_c \left( \alpha_t Y_t d_2 \left( 1 - \frac{\min[x_t, A_t]}{A_t} \right) - (N - S_t - x_t) \right) \]

Observe that for either simplified model by knowing \( \theta_t \) and observing \( S_t \) the firm can determine the exact (realized) values for \( \alpha_t \) and \( y_t \), and therefore at the sale time there is no uncertainty for the current period. If \( A_t + S_t = D(y_t) < N \), then the firm has excess capacity and bumping cannot occur. Otherwise the capacity is scarce, and bumping can occur if too many units are put on sale.

Since the firm knows which case realizes with certainty, it forces an intuitive restriction \( p_c \geq p_1 \). Otherwise firms could intentionally sell discounted products and later bump class-1 customers (as overflow) for a premium of \( p_1 - p_C > 0 \).
5.2. Single-Period Solution for the Simplified Models

Let \( \hat{x}(\alpha, y) \) be the maximum number of discounted units such that: (i) all class-2 customers are allocated a product without bumping others; and (ii) all \( \hat{x}(\alpha, y) \) units are sold. In the case of excess capacity the firm cannot sell all available inventory, and so \( \hat{x}(\alpha, y) = A(\alpha, y) \). For the scarce capacity case, solving \( \alpha yD_2 \left( 1 - \frac{x}{A(\alpha, y)} \right) = (N - S - x) \) yields \( \hat{x} = A(\alpha, y) \frac{N - S - \alpha yD_2}{\alpha(\alpha, y) - \alpha yD_2} < A(\alpha, y) \). Observe \( \hat{x} \leq N - S \). In summary,

\[
\hat{x}(\alpha, y) = \begin{cases} 
A(\alpha, y), & \text{if } A \leq N - S \text{ (excess capacity case)}; \\
A(\alpha, y) \frac{N - S - \alpha yD_2}{\alpha(\alpha, y) - \alpha yD_2}, & \text{if } A > N - S \text{ (scarce capacity case)}. 
\end{cases}
\]

Theorem 3: For a single period problem, in either simplified model, there exists a threshold waiting fraction \( \alpha^* \), such that if \( \alpha \geq \alpha^* \) then \( x^* = 0 \). Otherwise, in the case of scarce capacity, \( x^* = \hat{x}(\alpha, y) \), and in the case of excess capacity any \( x \in [\hat{x}(\alpha, y), N - S] \) is optimal.

We note that \( \alpha^* \leq 1 \) if \( D_t(Y_t)p_1 \leq Y_t d_2 p_2 \). That is, \( \alpha^* \leq 1 \) if the potential revenue at price \( p_2 \) exceeds the potential revenue at \( p_1 \). In this case, the single-period optimal policy is “bang-bang”: the optimal number of discounted units drops down to zero if too many customers wait (i.e., when \( \alpha \geq \alpha^* \)); and it jumps up to \( \hat{x} \) otherwise. Next we prove that a similar “bang-bang” policy holds for every period.

5.3. Multiple-Period Solution for Simplified Models

Let \( R_t(\theta_t, \alpha_t, y_t, x_t) \) be the expected revenue-to-go, given that for period \( t \), the waiting parameter is \( \theta_t \), the realized waiting fraction is \( \alpha_t \), the observed demand multiplier is \( y_t \) and \( x_t \) units are put on sale. The optimal number of discounted units, \( x^*_t \), can be found for each period, \( t = 1, 2, ... T \), by solving the following dynamic program:

\[
R_t(\theta_t, \alpha_t, y_t, x_t) = g_t(\alpha_t, y_t, x_t) + \delta E(\alpha_{t+1}, y_{t+1})|\theta_{t+1} = h_t(\theta_t, x_t) \left[ R^*_t(\theta_{t+1}, \alpha_{t+1}, y_{t+1}) \right]
\]

where the optimal revenue-to-go is given by

\[
R^*_t(\theta_t, \alpha_t, y_t) = \max_{0 \leq x_t \leq N - (1 - \alpha_t)y_t d_2} R_t(\theta_t, \alpha_t, y_t, x_t)
\]

where \( g_t(\alpha_t, y_t, x_t) \) is given by (9); \( R^*_t(\theta_{T+1}, \alpha_{T+1}, y_{T+1}) = 0 \) for all \( (\theta_{T+1}, \alpha_{T+1}, y_{T+1}) \); and \( \theta_{t+1} = h_t(\theta_t, x_t) \), is increasing and concave in either argument. Because \( \theta_{t+1} = h_t(\theta_t, x_t) \) does not depend on the realized values of \( \alpha_t \) and \( y_t \), we can write \( R_t(\theta_t, \alpha_t, y_t, x_t) = g_t(\alpha_t, y_t, x_t) + \delta \phi_{t+1}(h_t(\theta_t, x_t)) \).

In our two simplified models \( \phi_{t+1} \) takes the following specific forms:
In the deterministic waiting fraction (DW) model, $\alpha_t \equiv \theta_t$ for all $t$ by assumption and the distribution of $Y_t$ is independent of $\alpha_t$. Therefore
\[
\phi_{t+1}^{DW}(h_t(\alpha_t, x_t)) = E_{y_{t+1}}\left[R_{t+1}^*(h_t(\alpha_t, x_t), y_{t+1})\right]
\] (13)

In the deterministic demand (DD) model, $Y_t \equiv 1$ for all $t$ by assumption, and so w.l.o.g. $y$ can be dropped from the expectation of the future revenue, leading to
\[
\phi_{t+1}^{DD}(h_t(\theta_t, x_t)) = E_{(\theta_{t+1}, \alpha_{t+1})|h_t(\theta_t, x_t)}\left[R_{t+1}^*(\theta_{t+1}, \alpha_{t+1})\right]
\] (14)

Let $\Pi_t = \{x_t: R_t(\theta_t, \alpha_t, y_t, x_t) = R_t^*(\theta_t, \alpha_t, y_t) \text{ and } 0 \leq x_t \leq N - S_t\}$ be the set of “potentially optimal” solutions for period $t$. Our main result for the simplified models is that $\Pi_t = \{0; \hat{x}_t\}$ for all $t = 1, 2, \ldots T$. The concept of our proof is the following. Suppose that the expected future revenue, $\phi_{t+1}$, is decreasing and convex in $x_t$ and $\alpha_t$. Since $g_t(\alpha_t, x_t, y_t)$ is piecewise linear in $x$, $R_t$ consists of two adjacent and convex segments. Since $g$ is also decreasing for $x_t > \hat{x}_t$, $R_t$ is also decreasing if $x_t > \hat{x}_t$. Therefore $x^*_t \in \{0; \hat{x}_t\} \equiv \Pi_t$. We summarize this result in the theorem below.

**Theorem 4** In either simplified model, $x^*_t \in \Pi_t = \{0; \hat{x}_t\}$ for all periods $t = 1, 2, \ldots T$.

To summarize, for the case with a smoothing learning function for both simplified models, the optimal policy is “bang-bang”; it places either 0 or $\hat{x}_t$ units on the end-of-period sale depending on the realized waiting fraction, $\alpha_t$. The firm increases the number of “shoppers” by offering units on sale, and then withdraws revenue from those shoppers by periodically not offering any discounted units, subsequently decreasing the likelihood of future waiting. By following such policy the firm simultaneously achieves high utilization of its capacity and controls the number of customers waiting. This policy is quite different from that of the self-regulating case, because the firm actively manages the waiting, as opposed to relying on the consumers to regulate the waiting themselves. Observe that since $\hat{x}_t \leq N - S_t$ bumping is never optimal. This is because the marginal revenue $p_2$ per unit could as well be obtained from the initial sales, and since the future revenue is decreasing in $x_t$, the firm puts fewer units on sale and reduces the future waiting. This is not the case if the firm can obtain a marginal revenue in excess of $p_2$, as happens in the three price model that we study next.

**6. Three Price Model**

Next we study the case where a firm may choose to offer some units for sale at $p_1$, while raising the price to a higher value for the remaining inventory. By doing so the firm can both capture
the low-price demand, as well as the demand willing to pay extra for being accommodated after all discounted units are already sold. The three-price model reflects frequently observed situations where the “walk-up” price is higher than the regular, while some units have been sold at a discount earlier.

Let \( p_3 \geq p_2 \) be the “high” price. We build up on the simplified model of Section 5.1 amended as follows. We assume only class-3 customers are willing to purchase at price \( p_3 \) and let their demand in period \( t \) be \( Y_t d_3 \equiv Y_t D_3 \). The number of customers who are willing to pay price \( p_2 \) is now \( Y_t D_2 \) where \( D_2 = d_2 + d_3 \). Also let \( D_1 = d_1 + d_2 + d_3 \). We assume \( \bar{y} D_2 < N \) as before. At the sale time the firm decides \( x_t \), the number of units to offer at the discounted price \( p_1 \). The remaining units are offered at price \( p_3 \).

To determine the revenue of the firm, \( S_t \equiv (1 - \alpha_t) Y_t D_2 \) units are sold at the initial price \( p_2 \), and \( \min[x_t, A_t] \) units are sold at the discounted price \( p_1 \), where \( A_t \equiv (a + b Y_t) d_1 + \alpha_t Y_t D_2 \). Let \( \psi_t(\alpha_t) \in [0, 1] \) be the fraction of class-3 customers who wait for a discount, given that there is a fraction \( \alpha_t \in [\alpha; \bar{\alpha}] \) of class-2 and -3 customers waiting combined. That is, the total number of class-3 customers waiting is \( \psi_t(\alpha_t) Y_t d_3 \), and the number of class-2 customers is \( \alpha Y_t D_2 - \psi_t(\alpha_t) Y_t d_3 \). Since the latter is non-negative, it is implied that \( \psi_t(\alpha_t) Y_t d_3 \leq \alpha Y_t D_2 \) for all \( \alpha_t \in [\alpha; \bar{\alpha}] \).

As before, we assume that the discounted units are allocated on proportion. That is, the number of discounted units that are sold to class-3 customers is \( \min[x_t, A_t] \frac{\psi_t(\alpha_t) Y_t d_3}{A_t} \) and the net single-period revenue is

\[
g_t(\alpha_t, Y_t, x_t) = p_2 S_t + p_1 \min[x_t, A_t] + p_3 \psi_t(\alpha_t) Y_t d_3 \left( 1 - \frac{\min[x_t, A_t]}{A_t} \right)
- p_C \left( \psi_t(\alpha_t) Y_t d_3 \left( 1 - \frac{\min[x_t, A_t]}{A_t} \right) - (N - S_t - x_t) \right)^+, \]

where \( p_1 \leq p_C \leq p_3 \).

In this section we assume that the demand multiplier, \( Y_t \), is stochastic, and the waiting fraction, \( \alpha_t \), is deterministic.\(^2\) We assume that the distribution of \( Y_t \) does not depend on \( \alpha_t \), and that in turn, \( \alpha_t \) evolves according to a linear learning function \( \alpha_{t+1} = h_t(\alpha_t, x_t) \). We place no restriction whether \( h(\cdot) \) is smoothing or self-regulating.

In order to proceed we need to further specify the properties of \( \psi(\alpha) \). Consider the case where it is the firm’s policy not to oversell capacity and subsequently lose unsatisfied customer demand (the “no-bumping” case). Because class-3 customers have a higher valuation for the product, they

\(^2\) Alternative models with stochastic demand and waiting, or deterministic \( Y \) and stochastic \( \alpha \), could in general be of interest as well. However, we found that in three price context they place restrictive and uninterpretable conditions on prices, demands and waiting. Therefore we do not present them.
are less willing to wait and risk not receiving a unit. Therefore we assume $\psi$ is “small,” compared with $\alpha$, and class-3 customers do not wait unless many class-2 customers are already waiting. For example, if we assume that class-3 customers do not wait, unless all class-2 customers are already waiting, then $\psi(\alpha) = \max\{0, 1 - \frac{D_2}{d_3} + \alpha \frac{D_2}{d_3}\}$ for $\alpha \in [0, 1]$.

In the case where the firm is willing to bump passengers (the “bumping” case), class-3 customers who waited but did not get a discounted unit do not have capacity concerns, as they are guaranteed a product at their reservation price, $p_3$ (since $y_t d_3 \leq \bar{y} D_2 \leq N$). Therefore, they are more likely to wait, and we assume $\psi$ is “large”; there may be a fraction of class-3 customers waiting even if no class-2 customers wait. For example if $\psi(\alpha) = \frac{d_1 D_1}{A(\alpha, y)} + \alpha \frac{D_2}{D_2(D_1 - D_3)} - \frac{d_1 D_1}{A(\alpha, y)}$, for $\alpha \in \left(\frac{D_2}{D_2(D_1 - D_3)}, 1\right]$, the fraction of class-3 customers that always wait is $\psi(\alpha) = \frac{d_1}{d_1 + d_2}$.

We assume that $\psi$ is nondecreasing convex. In the no-bumping case $\psi = 0$ for $\alpha \leq \hat{\alpha} \leq d_2/D_2$, otherwise. In the bumping case, $\psi' D_3 \leq D_2$ and $\psi' A(\alpha, y) \leq \psi y D_2$. Functions that satisfy these assumptions correspond to the “small” and “large” $\psi$ as per the discussion above.

We redefine $\hat{x}$ to include the waiting of class-3 customers as follows:

\[
\hat{x}(\alpha, y) = \begin{cases} 
A(\alpha, y), & \text{if } A \leq N - S \text{ (excess capacity case)}; \\
A(\alpha, y) \frac{N - S - \psi(\alpha)yD_3}{A(\alpha, y) - \psi(\alpha)yD_3}, & \text{if } A > N - S \text{ (scarce capacity case)}. 
\end{cases}
\]

By the same argument as in the proof of Theorem 3 we obtain that the single-period optimal policy resembles that of the two-price model.

**Theorem 5** In the three price model there exists a threshold waiting fraction $\alpha^*$, such that if $\alpha \leq \alpha^*$, then $x^* = \hat{x}(\alpha, y)$ in the case of scarce capacity, and any $x \in [\hat{x}(\alpha, y), N - S]$ is optimal in the case of excess capacity. Otherwise, $x^* = 0$.

In this case, $\alpha^* \leq 1$ if $D_1(Y_t)p_1 \leq Y_t D_3 p_3$. That is, if the revenue from the high-price segment exceeds the revenue from all segments at the low-price, then the single-period optimal policy is “bang-bang.” For multiple periods the result extends as follows:

**Theorem 6** In the three-price model with bumping, $x_t^* \in \{0; \hat{x}_t; N - S_t\}$, and without bumping $x_t^* \in \{0; \hat{x}_t\}$ for all periods $t = 1, 2, \ldots, T$.

In sum, the optimal policy follows a pattern similar to that of the two-class model: increasing the fraction of customers waiting by offering units on sale followed by periods where no units are discounted. By doing so, the firm can withdraw revenue from waiting class-3 customers while decreasing future waiting. There is, however, a major difference between the two- and three-price
models. In the former, even if the firm’s policy was to bump customers if overflow occurs, doing so was never optimal. In contrast, in the latter, the optimal solution could involve selling all of the available \( N-S \) units at \( p_1 \), bumping some customers and paying penalty \( p_C < p_3 \). By doing so the firm can encourage more class-3 customers to wait, and potentially get revenue \( p_3 \) in excess of the regular price \( p_2 \) per unit, even though it may result in paying bumping penalties.

7. Numerical Studies

In this section we provide several examples to illustrate the value of making decisions optimally as compared with several heuristics managers use in real-life situations, and examine how this value and the optimal policy itself change in different situations. We also analyze how to select the optimal discount price \( p^*_1 \). We use the three price model since it allows for all types of consumer behavior that we study in our paper. We consider the case with positively correlated demands (i.e., with \( a = 0, b = 1 \) leading to the class-1 demand of \( Y_t d_1 \)).

We set \( N = 100, \delta = 0.95, D_2 = 50, p_1 = 100, p_2 = 300 \) and \( p_3 = 500, \) and consider four families of instances: smoothing bumping (MB), smoothing no-bumping (MN), self-regulating bumping (RB) and self-regulating no-bumping (RN). For each family of instances we study four demand curves, with \( D_1 = 150 \) or \( D_1 = 100 \), and \( D_3 = 30 \) or \( D_3 = 10 \) for class-3 customers. We denote these demand curves as “150-50-30,” “150-50-10,” “100-50-30,” and “100-50-10,” respectively. For cases with bumping we study two penalties: \( p_C = 150 \) and \( p_C = 450 \).

We use functions \( h(\alpha, x) = \lambda \frac{x}{N} + (1 - \lambda)\alpha \) and \( h(\alpha, x) = (1 - \lambda) + \lambda \frac{x}{N} - (1 - \lambda)\alpha \), for \( 0 < \lambda < 1 \) for the smoothing and self-regulating learning, respectively. We use \( \psi(\alpha) = \frac{D_1}{D_1} + \alpha \frac{D_2}{D_1} \) and \( \psi(\alpha) = \max[0; 1 - \frac{D_2}{D_1} + \alpha \frac{D_3}{D_1}] \) for waiting functions with and without bumping, respectively.

We set \( \bar{y} = 0.6 \) and \( \bar{y} = 1.4 \), so that the demand multiplier \( Y_t \in [0.6, 1.4] \), and examine three distributions: truncated Normal\([1, 0.2]^3\), Beta\([1.75, 3] \) and Beta\([0.8, 2] \), with the expected values and CVs, respectively, \((1, 0.913, 0.841)\) and \((0.181, 0.191, 0.237)\). The results are very similar and below we report only the case with Beta\(0.8;2) \).

For each instance we compute the expected infinite horizon discounted revenue (the revenue), assuming that the system starts from steady state – intuitively, this is the revenue that the firm will generate starting at an arbitrary time in the future. To compute the revenue we discretize \( \alpha_t \) as \( \{0; 0.01; 0.02; \ldots 1\} \) and discretize \( y_t \) as \( \{0.6; 0.7; \ldots 1.4\} \) for a total of 909 states. We use successive approximations with error bounds to compute the infinite horizon expected revenue function value (Bertsekas (1987), pp. 188-193). Then we determine the subset of recurrent states and the steady

\[^3\] We use \( 2\sigma \) limits so that the endpoints have non-negligible probabilities.
state probabilities, and obtain the expected revenue as the weighted sum (Puterman 1994, pp.589-594).

In Figure 2 (a) we present a sample path for the optimal decision, \( x^*_t \), and the fraction of customers waiting, \( \alpha_t \). In (b) we present the recurrent states and the frequencies with which they are visited in the steady state. We observe that the optimal decision and the fraction of customers waiting follow the cycles of variable length. This expresses the “bang-bang” structure of the optimal policy: if in period \( t \) the fraction of waiting customers, \( \alpha_t \), gets large enough, then the firm puts \( x^*_t = 0 \) units on sale and so \( \alpha_{t+1} \) drops. Hence in period \( t+1 \) the firm puts \( \hat{x} (\alpha_{t+1}, y_{t+1}) > 0 \) units on sale and \( \alpha_{t+2} \) increases again; note the 1-period time lag between \( x \) and \( \alpha \) in (a). Following such cycles, the optimal policy visits a variety of states; see (b). Because of the uncertainty in demand, the length of cycles is random. Thus, the customers cannot anticipate the transitions and hence the decision of the firm.

7.1. Performance of the Optimal Policy and Managerial Insights

To better understand the performance of the optimal policy we compare it with four heuristics that appeal to managers. In each heuristic we determine the number of units to put on sale through different methods. We consider the following:

Do-nothing: Let \( x_t = N - S_t \) or \( x_t = 0 \) for all \( t \), whichever is better;

BestP: Let \( x_t = N - S_t \) with probability \( P^* \) and \( x_t = 0 \) with probability \( 1 - P^* \), where the value of \( P^* \) is the one that results in the highest revenue;

\( S^* \): Let \( x_t = N - S \) if \( S_t \leq S^* \) and \( x_t = 0 \), otherwise, where the value of \( S^* \) is the one that results in the highest revenue;
Figure 3  The relative improvements for the (a) MN, (b) MB, (c) RN and (d) RB instances with 150-50-30 demand curve and Beta(0.8,2) demand multipliers. In (b) $p_C = 150$ and in (d) $p_C = 450$.

$\text{Beta}^*$: Let $x_t = N - S_t$ with probability $\beta^* \frac{N - S_t}{N}$, and $x_t = 0$ with probability $1 - \beta^* \frac{N - S_t}{N}$, where the value of $\beta^*$ is the one that results in the highest revenue.

The rationale behind the do-nothing heuristic is straightforward: choose the better of “all” or “none” on sale ad do so consistently in all periods. The BestP heuristic attempts to prevent consumers from guessing if a sale will occur in a given period.\(^4\) Heuristics $S^*$ and Beta* represent a naïve managerial approach where discounts are offered in periods with low regular price sales (i.e., when $S_t$ is small). The former heuristic does so when a threshold is crossed, whereas the latter places units on sale based on a linear probabilistic rule. We find $P^*$, $S^*$ and $\beta^*$ through numerical search. Given the revenues of the optimal and heuristic policies, we compute the relative improvement of the optimal policy over a particular heuristic as $(\text{optimal revenue} - \text{heuristic revenue}) ÷ \text{(heuristic revenue)}$.

\(^4\) A variation of the BestP heuristic is used by a car rental company with whom we discussed our work: they have a deal almost every week, but to access it, customers need a promotion code. These codes are e-mailed to a subset of their registered webmail customers, where a customer is included on the mailing list for a given week with some probability.
Instances where discounts improve revenue | Instances where discounts do no improve revenue
---|---
MB, RB 150-50-10 | MN, RN 150-50-10
MB, RB 100-50-10 | MN, RN 100-50-10
MN, RN 100-50-30 | MB, RB 100-50-30
MN, MB, RN, RB 150-50-30 |

Table 1 Comparison of cases where end-of-period discounts are beneficial.

Figure 3 presents the relative improvements over the heuristics for the four families of instances with 150-50-30 demand curve and Beta(0.8;2) demand multipliers. Our main observation is that in all cases the optimal policy generates five to fifteen percent additional revenue over the best heuristic. This value changes depending on the speed of learning and the type of consumer behavior. It also depends on which heuristic is the best.

In the cases without bumping, (Figure 3 (a) and (c)), the best heuristic is BestP, as it outperforms heuristics S* and Beta*. At a first glance this might seem slightly counterintuitive, since the latter are based on the intuitive managerial approach to put more units on sale when \( S_t \) is small. However, recall that the optimal policy suggest exactly the opposite to this naïve approach. Specifically, \( x_t^* = 0 \) if \( \alpha_t > \alpha^* \), and since \( S_t = (1 - \alpha_t) y_t D_2 \), it follows that (in expectation) it is optimal not to put units on sale in the periods with small \( S_t \).

The improvement over the BestP heuristic depends on the speed of learning. This is because the optimal policy determines when to offer a discount (the timing), and if one is offered, then how many units to discount (the number). By choosing the best probability, the BestP heuristic “optimizes” the long-run average number of units on sale, but cannot achieve the right timing of sales. In the cases of slow learning in order to change waiting behavior, the firm must have consistent series of periods with and without discounts. The BestP heuristic cannot ensure such consistency, and therefore chooses to do nothing (indeed \( P^* = 0 \) for \( \lambda < 0.625 \) on Figure 3 (a) and for \( \lambda < 0.575 \) on Figure 3 (c)). For faster speeds of learning, consistency is not required as customers readily change their waiting behavior, and therefore the timing of sales is less important than the average number of units on sale.

In the cases with bumping, i.e., where \( p_C < p_3 \), (Figure 3 (b) and (d)), the best heuristic is S*. The S* heuristic discounts units when demand is likely to be low. In contrast, the BestP and Beta* can place units on sale when demand is high and thus incur bumping penalties.

Next we study the factors that influence whether strategic revenue management as we discuss in this paper will be effective. Table 1 classifies different instances into those where the firm benefits from offering end-of-period discounts and those where it does not. In the cases with few class-3 customers (rows 1 and 2 in Table 1) observe that discounts increase revenue only in the cases with
bumping. This is because if the firm bumps customers to accommodate class-3 overflow, then it provides an incentive for more class-3 customers to wait (through the $\psi(\alpha)$ function). But because there are few class-3 customers overall, the number of waiting class-3 customers is still small compared to the number of class-1 customers. As a result, only a few class-3 customers who wait are able to buy at a discount, while most buy at $p_3$, hence increasing firm’s revenue. In contrast, when there are many class-3 customers and few class-1 customers (row 3), the threat imposed by refusing to bump customers forces fewer high-value class-3 customers to wait (through $\psi(\alpha)$), allowing the firm to sell much of its capacity early at price $p_2$ and gain revenue by offering discounts that are mostly purchased by class-1 customers. Without the threat (i.e., with bumping) the reaction function $\psi(\alpha)$ changes and too many class-3 customers wait and purchase discount seats, instead of purchasing them early at price $p_2$. Thus, given a choice, depending on the relative class-1 and class-3 demand, the firm may or may not prefer to oversell its capacity and bump passengers. That is, the firm can use bumping strategically to induce appropriate customer behavior and enhance the effectiveness of its revenue management policy.

7.2. Selecting the Optimal Discount Price, $p_1^*$

Our model assumes that the discount price, $p_1$, is fixed for the entire horizon of $T$ periods, and as we argue in the introduction, building a model where customers react to both price and availability of discounted units is nontrivial. As research in dynamic pricing shows, however, a heuristic that charges an optimally selected single price (as opposed to optimizing it dynamically) often performs only marginally suboptimally. Therefore, as a heuristic policy, the firm could search for the optimal “static” discounted price, $p_1^*$, charge it in every period, and then determine the number of discounted units to put on sale following our optimal policy.

We search for such optimal static price, $p_1^*$, numerically over its domain, $[0, p_2]$; see Figure 4. In this example, we impose a demand curve $D(p) = 200 - 0.5p$ (so that $D(300) = 50$ and $D(100) = 150$ as in previous examples). We also assume that the value of a discount influences the rate at which customers are willing to change their behavior, i.e., the speed of learning. In particular we assume $\lambda(p) = 0.3 - 0.001p$. We experimented with other functions for demand and speed of learning, but observed no qualitative differences from the case presented.

Two observations are evident from Figure 4. First, the revenue is not concave and often not quasiconcave in the price, therefore it may not be possible to determine the optimal discount analytically. There is an obvious optimal point, however. Such a point exists because the firm uses class-1 demand to achieve two goals. On one hand it wants $p_1$ to be high, since then it obtains
larger revenue from the sale of each discounted unit. On the other hand, it wants $D(p_1)$ to be high, because under proportional allocation class-1 customers displace some waiting class-3 customers, so that they purchase at the higher price, $p_3$. Since price negatively affects demand, these goals conflict and an optimal trade-off point is found.

Figure 4 also provides a neat illustration to the earlier point that firms can strategically use bumping to increase revenue. Observe that when the discounted price is small, hence, class-1 demand is high, more revenue is obtained when the firm bumps passengers (RB and MB curves are higher than RN and MN). Conversely, when the price is large, class-1 demand is small, and the firm benefits from not bumping (RN and MN curves are higher).

8. Conclusions

Our work is motivated by the concern that given the increased ability to search for better prices for travel related products (flights, vacation packages, etc.), consumers will learn to expect end-of-period deals and will strategically wait for them. We study the problem for cases of two and three customer classes. The two-class problem represents the case where a list price is given (as in the cruise or vacation packages industries); the three-class problem reflects typical airline pricing where prices may decrease or increase in the days prior to departure. We formulate the problem as a dynamic program and develop a unique solution approach amenable to the novel structural properties we find in the problem.

For the case of two customer classes with self-regulating customer behavior, we show that the firm in general will set some units on sale in each period and allow the customer behavior to limit the number receiving the benefit of the reduced price inventory. In contrast, in the case of smoothing customer behavior, we show the firm should follow a “bang-bang” sale policy, either
placing most of the remaining units on sale or none. Thus, the firm takes a more active role, adjusting the customers’ expectations by alternately increasing the number of customers waiting until a threshold is crossed, upon which the firm places no units on sale. By doing so, the firm is able to regulate the number of customers waiting and to increase its revenue by increasing utilization, allowing some units that would otherwise not be sold to be purchased by the lower-value customers.

In the model with three customer classes, we consider how high-value customers react to a firm’s bumping policy. In the cases with bumping, high-value customers have a higher incentive to wait. As a result, in addition to the “bang-bang” policy, it could be optimal to discount all units remaining and pay bumping penalty. Such policy, however, is beneficial only if there are few high-value customers. Establishing a policy of not bumping passengers is beneficial when there are many high-value customers. Overall, the benefit from following the optimal policy is 5 to 15 percent more revenue as compared to several intuitive managerial heuristics.

We acknowledge that the model formulated here does not account for all factors that may influence strategic customer behavior in revenue management. Future studies should consider more explicit formulations of customer utility, competition between firms and gaming by both firms and customers. In addition, empirical work is needed to better understand consumer learning behavior with regards to travel-related discounts.

References


Appendix A: Proofs

We require the following fundamental theorem:

**Theorem 7 (3.9.1 in Topkis 1998)** If $T$ is a subset of $\mathbb{R}^m$, $\{F_X(x;\theta) : \theta \in T\}$ is a collection of distribution functions, and $\mathcal{F}$ is a closed (in the topology of pointwise convergence), convex cone of real-valued functions on $T$, then for any increasing set $S$, $\int_S dF_X(x;\theta)$ is in $\mathcal{F}$ iff $\int_S v(x)dF_X(x;\theta)$ is in $\mathcal{F}$ for any increasing real-valued function $v(x)$.

We also require the following two results:

**Proposition 1 (follows from Theorem 5.3 in Rockafellar 1997)** If $f(x,\theta)$ is jointly concave in $(x,\theta)$, then $\sup_x f(x,\theta)$ is concave in $\theta$.

**Proposition 2 (Theorem 2.7.6 in Topkis 1998)** If $f(x,\theta)$ is supermodular in $(x,\theta)$, then $\sup_x f(x,\theta)$ is supermodular in $\theta$.

**Proof of Lemma 1.** Recall that by assumption (A5) $h$ is linear.

(i) Concavity follows from

$$\frac{\partial^2}{\partial x^2} J(\theta,S,x) = \frac{\partial^2}{\partial x^2} r(\theta,S,x) + \delta \frac{\partial^2}{\partial h^2} \left( \frac{\partial h}{\partial x} \right)^2 \leq 0$$

since $r$ is concave in $x$ by assumption (A1) and $\phi$ is concave in $h$ by the condition of the lemma.

(ii) Supermodularity in $(\theta,x)$ follows from

$$\frac{\partial^2}{\partial x \partial \theta} J(\theta,S,x) = \frac{\partial^2}{\partial x \partial \theta} r(\theta,S,x) + \delta \frac{\partial^2}{\partial h^2} \frac{\partial h}{\partial x} \frac{\partial h}{\partial \theta} \geq 0$$

since $r$ is supermodular in $(\theta,x)$ by assumption (A2), $\phi$ is concave in $h$ by the condition of the lemma and $h$ is self-regulating by assumption (A5).

(iii) Supermodularity in $(\theta,S)$ and $(S,x)$ follows from (A2) because $\phi$ does not depend on $S$. Supermodularity in multiple dimensions is equivalent to supermodularity in each pair (Topkis (1998), Theorem 2.6.1). Q.E.D.

**Proof of Lemma 2.** Let $\hat{\theta}$ be an arbitrary fixed value of $\theta$.

Then

$$\left. \frac{\partial^2}{\partial \theta^2} \left( \int v(\theta,x)dF_X(x,\theta) \right) \right|_{\theta=\hat{\theta}} = \left. \frac{\partial}{\partial \theta} \left( \int f_X(x;\theta)d\theta \right) \right|_{\theta=\hat{\theta}} + \int v(\theta,x) \left. \frac{\partial}{\partial \theta} f_X(x;\theta) \right|_{\theta=\hat{\theta}} d\theta$$

$$= \int \left. \frac{\partial^2}{\partial \theta^2} v(\theta,x) \right|_{\theta=\hat{\theta}} f_X(x;\theta) d\theta + \int \left. v(\theta,x) \frac{\partial}{\partial \theta} f_X(x;\theta) \right|_{\theta=\hat{\theta}} d\theta$$

$$+ \int \left. v(\theta,x) \frac{\partial^2}{\partial \theta^2} f_X(x;\theta) \right|_{\theta=\hat{\theta}} dx$$
\[
\begin{align*}
&\leq 2 \int \left( \frac{\partial}{\partial \theta} v(\theta, x) \right) \left( \frac{\partial}{\partial \theta^2} f_x(x; \theta) \right) |_{\theta = \hat{\theta}} dx + \int v(\hat{\theta}, x) \left( \frac{\partial^2}{\partial \theta^2} f_x(x; \theta) \right) |_{\theta = \hat{\theta}} dx \\
&\leq \int v(\hat{\theta}, x) \left( \frac{\partial^2}{\partial \theta^2} f_x(x; \theta) \right) |_{\theta = \hat{\theta}} dx \\
&= \frac{\partial^2}{\partial \theta^2} \int v(\hat{\theta}, x) dF_x(x; \theta) \\
&\leq 0
\end{align*}
\]

The first inequality follows from the concavity of \(v(\theta, x)\) in \(\theta\). The second inequality follows by Theorem 7 because \(v(\theta, x)\) is supermodular (i.e. \(\partial v/\partial \theta\) is increasing in \(x\)), while \(X_\theta\) is stochastically decreasing. Finally, the third inequality results from Theorem 7 because \(v(\theta, x)\) is increasing in \(x\) and \(X_\theta\) is stochastically concave. Q.E.D.

**Proof of Theorem 1.** For \(t = T\) the claim holds by assumptions (A1) and (A2) respectively. Let \(1 \leq t \leq T\) and suppose that for every period \(n \in [t+1, T]\): (I1) \(J^n_\ast(\theta_n, S_n)\) is increasing in \(S_n\); (I2) \(J^n(\theta_n, S_n)\) is concave in \(\theta_n\) and (I3) \(J^n(\theta_n, S_n)\) is supermodular in \((\theta_n, S_n)\).

With assumptions (A4) and (I1)-(I3) following Lemma 2, \(\phi_{t+1}\) is concave in \(h_t\), and therefore with assumptions (A1), (A2) and (A5) by Lemma 1, \(J_t(\theta_t, S_t, x_t)\) is concave in \(x_t\) and supermodular in \((\theta_t, S_t, x_t)\).

For period \(t\): (I1) follows from (A3) since \(\phi_{t+1}\) does not depend on \(S_t\); (I2) follows by Proposition 1 from (A1), since (A5) implies that the Hessian of \(\phi_{t+1}\) equals \(\frac{\partial^2 \phi}{\partial h^2} \left( \frac{\partial h}{\partial x} \right)^2 \frac{\partial^2 \phi}{\partial x^2} \left( \frac{\partial h}{\partial \theta} \right)^2 - \left( \frac{\partial^2 \phi}{\partial h^2} \left( \frac{\partial h}{\partial x} \right) \frac{\partial^2 \phi}{\partial x \partial \theta} \right)^2 = 0; (I3) follows by Proposition 2 since \(J_t(\theta_t, S_t, x_t)\) is supermodular in \((\theta_t, S_t, x_t)\). Q.E.D.

**Proof of Lemma 3.** Let \(y_L(S, x) = \{y : D(y) \leq S + x\}\) and let \(y_H(S, x) = \{y : B(S, x, y) > N - S - x\}\). Thus, \(y_L(S, x)\) is the set of demand outcomes where all demand is accommodated prior to any last minute price increase; \(y_H(S, x)\) is the set of outcomes where unsatisfied class-2 demand exceeds the remaining capacity at this time and thus customers will be bumped (if \(p_C < p_2\)). Note that by (B4), \(y_L(S, x) \cap y_H(S, x) = \emptyset\).

By the definition of \(y_L(S, x)\) and \(y_H(S, x)\):

\[
g(S, x, \hat{y}) = \begin{cases} 
p_2 S + p_1(D(\hat{y}) - S), & \text{if } \hat{y} \in y_L(S, x); \\
p_2 S + p_1 x + p_2 B(S, x, \hat{y}), & \text{if } \hat{y} \notin y_L(S, x) \cup y_H(S, x); \\
p_2 S + p_1 x + (p_2 - p_C) B(S, x, \hat{y}) + (N - S - x)p_C, & \text{if } \hat{y} \in y_H(S, x). 
\end{cases}
\]

By assumption (B1), \(g(S, x, \hat{y})\) is non-decreasing in \(\hat{y}\). Similarly, since \(p_2 > p_1\), by assumption (B3), \(g(S, x, \hat{y})\) is non-decreasing in \(S\).

Let \(f_Y(y; S)\) be the pdf of \(\hat{Y}\), \(F_Y(y; S)\) be the CDF and let \(\hat{S}\) be an arbitrary fixed value of \(S\). Then \(r(\theta, S, x)\) is increasing in \(S\) because

\[
\frac{\partial}{\partial S} \left( \int g(S, x, y) dF_Y(y; S) |_{S = \hat{S}} \right)
\]
= \int \frac{\partial}{\partial S} (g(S,x,y) f(y;S)) |_{S=S} dy \\
= \int \left( \frac{\partial}{\partial S} g(S,x,y) \right) |_{S=S} f_Y(y) dy + \int g(S,x,y) \left( \frac{\partial}{\partial S} f_Y(y;S) \right) |_{S=S} dy \\
\geq \int g(\hat{S},x,y) \left( \frac{\partial}{\partial S} f_Y(y;S) \right) |_{S=S} dy \\
= \frac{\partial}{\partial S} \int g(\hat{S},x,y) dF_Y(y;S) \\
\geq 0

The first inequality holds because \( g(S,x,y) \) is nondecreasing in \( S \). The second holds by Theorem 7 because \( g(S,x,y) \) is increasing in \( \hat{y} \) while \( \tilde{Y} \) is stochastically increasing in \( S \).

Similarly, \( r(\theta,S,x) \) is concave in \( \theta \) by Theorem 7 because \( g(S,x,y) \) is nondecreasing in \( \hat{y} \) and \( \hat{y} \) is stochastically concave in \( \theta \). Q.E.D.

**Proof of Lemma 4.** Let \( \hat{x} \) be the solution to \( B(S,\hat{x},\hat{y}) = N - S - \hat{x} \). Since \( B \) is decreasing in \( x \) and \( \partial B/\partial x \geq -1 \), then \( B(S,x,\hat{y}) \leq N - S - x \) if \( 0 \leq x \leq \hat{x} \), and \( B(S,x,\hat{y}) > N - S - x \) if \( \hat{x} < x \leq N - S \). Because \( B(S,x,\hat{y}) \) is nonnegative, \( \hat{x} \leq N - S \).

Consider two cases:

**Case 1:** if \( D(\hat{y}) > N \) then \( x \leq N - S \leq D(\hat{y}) - S \). So from (1) we obtain

\[
g(S,x,\hat{y}) = \begin{cases} 
p_2 S + p_1 x + p_2 B(S,x,\hat{y}), & \text{if } 0 \leq x \leq \hat{x}; \\
(p_2 - p_C)S + (p_1 - p_C)x + (p_2 - p_C)B(S,x,\hat{y}) + p_CN, & \text{if } \hat{x} < x \leq N - S.
\end{cases}
\]

Noting \( g(S,x,\hat{y}) \) is continuous at \( \hat{x} \), concave on both segments and \( \lim_{x \uparrow \hat{x}} \frac{\partial g}{\partial x} = \lim_{x \downarrow \hat{x}} \frac{\partial g}{\partial x} + p_C(1 - \frac{\partial B}{\partial x}) \), \( g(S,x,\hat{y}) \) is concave by assumption (B2) and \( p_C \geq 0 \).

**Case 2:** if \( D(\hat{y}) \leq N \) then \( S + x + B(S,x,\hat{y}) \leq N \). By (1)

\[
g(S,x,\hat{y}) = \begin{cases} 
p_2 S + p_1 x + p_2 B(S,x,\hat{y}), & \text{if } 0 \leq x \leq D(\hat{y}) - S; \\
p_2 S + p_1 (D(\hat{y}) - S), & \text{if } D(\hat{y}) - S < x \leq N - S.
\end{cases}
\]

\( g(S,x,\hat{y}) \) is continuous and if \( \partial B/\partial x \geq -p_1/p_2 \), then \( g(S,x,\hat{y}) \) is non-decreasing in \( x \), and by assumption (B5) is concave.

Because concavity is maintained under expectation over an exogenous random variable, \( r(\theta,S,x) \) is concave in \( x \). Q.E.D.

**Proof of Corollary 1.** We use the definitions of \( y_L(S,x) \) and \( y_H(S,x) \) given in Lemma 3. From (1) and (2) we obtain

\[
r(\theta,S,x) = \int_{y_L(S,x)} (p_2 S + p_1 (D(y) - S)) dF_Y(y) \\
+ \int_{y_L(S,x) \cup y_H(S,x)} (p_2 S + p_1 x + p_2 B(S,x,y)) dF_Y(y) \\
+ \int_{y_H(S,x)} ((p_2 - p_C)S + (p_1 - p_C)x + (p_2 - p_C)B(S,x,y) + p_CN) dF_Y(y)
\]
where the limits of the integration follows from the definitions of \(y_L\) and \(y_H\). Differentiating \(r(\theta, S, x)\) in \(x\) we obtain

\[
\frac{\partial r}{\partial x} = \int_{Y \setminus (y_L \cup y_H)} \left( p_1 + p_2 \frac{\partial B(S, x, y)}{\partial x} \right) dF_Y(y) + \int_{y_H} \left( (p_1 - p_C) + (p_2 - p_C) \frac{\partial B(S, x, y)}{\partial x} \right) dF_Y(y).
\]

Setting \(\partial r / \partial x = 0\) and rearranging the terms gives (5) Q.E.D.

**Proof of Lemma 5.**

For joint concavity, observe that since \(r\) is concave in \(\theta\) and in \(x\) by lemmas 3 and 4, respectively, it is sufficient to show that the determinant of the Hessian \(\partial^2 r / \partial x^2 \partial^2 \theta^2 - (\partial^2 r / \partial x \partial \theta)^2 > 0\).

For supermodularity, observe that by Theorem 7 if \(g(S, x, \hat{y})\) is supermodular in \((S, x, \hat{y})\), and because \(\hat{Y}\) is stochastically supermodular in \((\theta, S)\), then \(r(\theta, S, x)\) is also supermodular in \((\theta, S, x)\). (See Section 3.10.1 in Topkis (1998)).

Consider two cases:

**Case 1:** if \(D(\hat{y}) > N\) then \(x \leq N - S \leq D(\hat{y}) - S\). From (1) we obtain

\[
g(S, x, \hat{y}) = \begin{cases} 
  p_2 S + p_1 x + p_2 B(S, x, \hat{y}), & \text{if } B - (N - S - x) < 0; \\
  (p_2 - p_C) S + (p_1 - p_C) x + (p_2 - p_C) B + p_C N, & \text{if } B - (N - S - x) \geq 0.
\end{cases}
\]

If \(p_C = 0\) then by the definition of \(B\) and the conditions of the Lemma, \(B = Q^2(\hat{y}) - S - cx\) for some constant \(c > 0\). Thus \(g = p_2 Q^2(\hat{y}) + x(p_1 - cp_2)\), which is clearly supermodular. If \(p_C > 0\) then by the conditions of the lemma \(B = Q^2(\hat{y}) - S - x\), and thus \(B - (N - S - x) < 0\) iff \(Q^2(\hat{y}) < N\). Since by definition \(Q^2(\hat{y}) < N\), it therefore follows that \(g = p_2 Q^2(\hat{y}) + (p_1 - p_2) x\) which is clearly supermodular. In either case \(g\) is linear in \((x, \hat{y})\). Thus \(\frac{\partial r}{\partial x} = (p_1 - cp_2) \int_y dF_{\theta,S} = \text{const.}\), and so \(\frac{\partial^2 r}{\partial x \partial \theta} = 0\), i.e., \(r\) is jointly concave.

**Case 2:** if \(D(\hat{y}) \leq N\) then

\[
g(S, x, \hat{y}) = \begin{cases} 
  p_2 S + p_1 x + p_2 B(S, x, \hat{y}), & \text{if } 0 \leq x \leq D(\hat{y}) - S; \\
  p_2 S + p_1 (D(\hat{y}) - S), & \text{if } D(\hat{y}) - S < x \leq N - S.
\end{cases}
\]

By the conditions of the lemma \(B = Q^2(\hat{y}) - S - \frac{p_3}{p_2} x\). Thus \(g = p_2 Q^2(\hat{y})\) for \(0 \leq x \leq D(\hat{y}) - S\), and \(g = p_2 S + p_1 (D(\hat{y}) - S)\) for \(D(\hat{y}) - S \leq x \leq N - S\). Further, by condition (B4), \(B = 0\) at \(x = D(\hat{y}) - S\), i.e., \(p_2 Q^2(\hat{y}) = p_2 S + p_1 (D(\hat{y}) - S)\). This implies that the values of \(g\) are equal on both intervals and so \(g\) is independent of \(x\) and therefore supermodular and jointly concave.

Q.E.D.

**Proof of Theorem 3.** In the case of excess capacity if \(x > \hat{x} = A\), then \(g(\alpha, y, x) = p_2 (1 - \alpha) y D_2 + p_1 A\) which is independent of \(x\).

In the case of scarce capacity by definition \(A > N - S \geq x\). If \(x > \hat{x}\), then \(g(\alpha, y, x) = p_2 S + p_1 x + (p_2 - p_C) \alpha y D_2 (1 - \frac{x}{A}) + p_C (N - S - x)\), and so \(\frac{\partial g}{\partial x} = (p_1 - p_C) - (p_2 - p_C) \frac{\alpha y D_2}{x^2} \leq 0\) as \(p_C \geq p_1\).
Therefore, \( x^* \in [0, \hat{x}(\alpha, y)] \). On this interval \( g(\alpha, y, x) = p_2 S + p_1 x + p_2 \alpha y D_2 \left( 1 - \frac{x}{\hat{x}} \right) \), which is linear in \( x \), and so the optimal solution in on the boundary of the interval; that is \( x^* \in \{0; \hat{x}(\alpha, y)\} = \Pi_1 \).

We next show the existence and uniqueness of the threshold waiting fraction. Let \( C(\alpha) = \frac{\partial g}{\partial x} \) and observe \( C(\alpha) = p_1 - \frac{\alpha y p_2 D_2}{(\alpha + b y) d_1 + \alpha y D_2} \). Differentiating it we obtain \( \frac{\partial^2 C}{\partial x^2} = -\frac{p_2 D_2 d_1 Y (\alpha + b y)}{(\alpha + b y d_1 + \alpha y D_2)^2} \leq 0 \). Therefore \( \alpha^* \) solves \( C(\alpha^*) = 0 \). Since \( C(0) = p_1 > 0 \) and \( C \) is monotonically decreasing, \( \alpha^* \) is unique.

Finally, for \( \alpha \geq \alpha^* \), \( C(\alpha) \leq 0 \); that is \( g(\alpha, x, y) \) is nonincreasing in \( x \) and so \( x^* = 0 \). Otherwise the maximal revenue is attained at \( x^* = \hat{x}(\alpha, y) \). In the case of excess capacity, however, the revenue function is flat on \([\hat{x}(\alpha, y), N - S] \), and so in this case any \( x \in [\hat{x}(\alpha, y), N - S] \) is optimal if \( \alpha \leq \alpha^* \).

Q.E.D.

**Proof of Theorem 4.** Recall that we consider two simplified models: the one with a deterministic waiting fraction, given by (13), and the one with deterministic demand, given by (14). We first prove the theorem for the deterministic waiting case, and then extend it to deterministic demand case. We require the following three lemmas.

**Lemma 6** \( g(\alpha, y, x) \) is decreasing convex in \( \alpha \) for \( x \in \Pi_1 \).

**Proof** If \( x = 0 \) then \( g(\alpha, y, 0) = p_2 y (1 - \alpha) D_2 + p_2 \alpha y D_2 = p_2 y D_2 \), which is not a function of \( \alpha \), and therefore is decreasing convex in a weak sense.

If \( x = \hat{x} \) then in the excess capacity case \( g(\alpha, y, \hat{x}(\alpha, y)) = p_2 (1 - \alpha) y D_2 + p_1 ((a + by)d_1 + \alpha y D_2) \) which is linear decreasing in \( \alpha \) since \( \frac{\partial g}{\partial \alpha} = -y D_2 (p_2 - p_1) \leq 0 \) because \( p_2 \geq p_1 \). In the scarce capacity case observe that \( \hat{x} = \frac{N - S - \alpha y D_2}{(a + by) d_1 + \alpha y D_2} = \frac{N - S - \alpha y D_2}{(a + by) d_1} \), which is linear in \( \alpha \). Therefore \( g(\alpha, y, \hat{x}(\alpha, y)) = p_2 y (1 - \alpha) D_2 + p_1 x + (p_2 - p_1) \frac{N - S - \alpha y D_2}{(a + by) d_1} + p_c (N - S - x) \), which is linear decreasing in \( \alpha \) since \( \frac{\partial g}{\partial \alpha} = -\frac{y D_2 (p_2 - p_1) (N - \hat{y} D_2)}{(a + by) d_1} \leq 0 \) as \( y D_2 \leq \hat{y} D_2 \leq N \) and \( p_2 \geq p_1 \) by the assumption, and \( \frac{\partial^2 g}{\partial \alpha^2} = 0 \). Q.E.D.

**Lemma 7** If \( f(x) \) is decreasing convex and \( g(x) \) is increasing concave, then \( f(g(x)) \) is decreasing convex.

**Proof** Follows by the chain rule.

Convex: \( \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} \left( \frac{\partial g}{\partial x} \right)^2 + \frac{\partial f}{\partial y} \frac{\partial^2 g}{\partial x^2} \geq 0 \), because \( f \) is decreasing convex and \( g \) is concave.

Decreasing: \( \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} \frac{\partial g}{\partial x} \leq 0 \), because \( f \) is decreasing, and \( g \) is increasing. Q.E.D.

**Lemma 8** Let \( f(x, y) \) be decreasing and(or) convex in \( x \) for all \( y \in Y \), then

(a) \( \sum_{y \in Y} f(x, y) \) is decreasing and(or) convex in \( x \);

(b) \( \sup_{y \in Y} f(x, y) \) is decreasing and(or) convex in \( x \);
Proof Follows from Theorems 5.2 and 5.5 in Rockafellar (1970). Q.E.D.

(i) Deterministic waiting model. For period $T$ the claim is implied by Theorem 3. For $x_T \in \Pi_T$, by Lemma 6, the single-period revenue, $g_T(\alpha_T, y_T, x_T)$, is decreasing and convex in $\alpha_T$. Therefore since $R^*_T(\theta_T, \alpha_T, y_T) = \max_{x_T \in \Pi_T} g_T(\alpha_T, y_T, x_T)$, $R^*_T$, is also decreasing and convex in $\alpha_T$ by part (b) of Lemma 8.

In the deterministic waiting model $\alpha_T \equiv \theta_T = h_{T-1}(\theta_{T-1}, x_{T-1})$, and so by the above $R^*_T$ is also decreasing and convex in $h_{T-1}(\cdot)$. Therefore its expectation over $y_T$, $\phi^T_{DW}$ as per (13), is decreasing and convex in $h_{T-1}(\cdot)$ by part (a) of Lemma 8.

Let $1 \leq t < T$ and suppose that for every $n \in [t+1, T]$, $\phi_n(h_{n-1}(\theta_{n-1}, x_{n-1}))$ is decreasing convex in $h_{n-1}$. Then by Lemma 7, in period $t+1$, $\phi_{t+1}$ is decreasing and convex in $\theta_t$ and $x_t$.

Recall that $g_t(\alpha_t, y_t, x_t)$ is piecewise linear in $x_t$ on $[0, N-S_t]$ with the breakpoint at $x_t = \hat{x}_t$, and further recall that $g_t(\alpha_t, y_t, x_t)$ is decreasing in $x_t$ on $x_t \in (\hat{x}_t, N-S_t]$. Since $\phi_{t+1}$ is decreasing and convex in $x_t$, $R_t(\theta_t, \alpha_t y_t, x_t)$ consists of two adjacent segments both convex in $x_t$, and $R_t(\theta_t, \alpha_t y_t, x_t)$ decreases in $x_t$ on $x_t \in (\hat{x}_t, N-S_t]$. Thus $x^*_t \in [0, \hat{x}_t]$. Finally, since the revenue-to-go function is convex on this interval, $x^* \in \{0; \hat{x}_t\} \equiv \Pi_t$.

Therefore it is sufficient to prove that the induction assumption holds for period $t$; that is that $\phi_t$ is decreasing and convex in $h_{t-1}$.

For $x_t \in \Pi_t$, the single-period revenue, $g_t(\alpha_t, y_t, x_t)$, is decreasing and convex in $\alpha_t$ by Lemma 6. The future revenue, $\phi_{t+1}$, is also decreasing and convex in $\alpha_t$ by the induction assumption, upon noting that in the deterministic waiting model $\alpha_t \equiv \theta_t$. Therefore $R_t(\theta_t, \alpha_t y_t, x_t)$ is decreasing and convex in $\alpha_t$. And so by part (b) of Lemma 8, $R^*_t$ is also decreasing and convex in $\alpha_t$. Finally, since in deterministic waiting model $y_t$ is independent of $\alpha_t$, by part (a) of Lemma 8 $\phi_t = E_{y_t}[R^*_t]$ is decreasing and convex in $\alpha_t$, and therefore in $h_{t-1}$ (since $\alpha_t \equiv \theta_t = h_{t-1}$).

(ii) Deterministic demand model. For period $T$ the claim is implied by the single-period result, given in Theorem 3.

In the deterministic demand model, observe that $R^*_T(\theta_T, \alpha_T, 1) = \max_{x_T \in \Pi_T} g(\alpha_T, 1, x_T)$ is independent of $\theta_T$. Thus, from (14), $\phi^T_{DW} = E_{\alpha_T[h_{T-1}]}[R^*_T(\alpha_T)]$, which is decreasing and convex in $h_{T-1}$ by Theorem 7, since by the above $R^*_T$ is decreasing in $\alpha_T$, and $\alpha_T$ is stochastically increasing and concave in $\theta_T$ by our assumption. Therefore by Lemma 7, $\phi_T$ is decreasing and convex in $\theta_{T-1}$ and $x_{T-1}$, since $h_{T-1}(\theta_{T-1}, x_{T-1})$ is increasing and concave.

Let $1 \leq t < T$ and suppose that for every $n \in [t+1, T]$, $\phi_n(h_{n-1}(\theta_{n-1}, x_{n-1}))$ is decreasing convex in $h_{n-1}$. Then by Lemma 7, in period $t+1$, $\phi_{t+1}$ is decreasing and convex in $\theta_t$ and $x_t$. With this
\( x_t^* \in \{0; \hat{x}_t\} \equiv \Pi_t \) by the same argument as in the proof of Theorem 4, and it remains to prove that \( \phi_t \) is decreasing and convex in \( h_t-1 \).

Since \( \alpha_t \) is stochastically increasing and concave in \( \theta_t = h_{t-1}(\cdot) \), the vector \((\theta_t, \alpha_t)\) is stochastically increasing and concave in \( h_{t-1} \).

For \( x_t \in \Pi_t \), single-period revenue \( g_t(\alpha_t, x_t) \), is decreasing in \( \alpha_t \) by Lemma 6 and independent of \( \theta_t \). Future revenue, \( \phi_{t+1} \), is independent of \( \alpha_t \), and is decreasing in \( h_t \) by the induction assumption, and therefore by Lemma 7 \( \phi_{t+1} \) is also decreasing in \( \theta_t \). Thus \( R_t(\theta_t, \alpha_t, y, x_t) \) is decreasing in \((\theta_t, \alpha_t)\). And by part (b) of Lemma 8, \( R_t^* \) is also decreasing in \((\theta_t, \alpha_t)\). From (14) \( \phi_t = \frac{E(\theta_t, \alpha_t)}{R_t^*} \) is decreasing and convex in \( h_{t-1}(\cdot) \) by Theorem 7 because \((\theta_t, \alpha_t)\) is stochastically increasing and concave in \( h_{t-1} \). Q.E.D.

**Proof of Theorem 6.** Follows by the same argument as in the deterministic waiting case in the proof of Theorem 4. As in Lemma 6 we prove that \( g_t(\alpha_t, x_t) \) is convex in \( \alpha_t \) for \( x_t \in \Pi_t \), where \( \Pi_t \) is redefined as \( \{0, \hat{x}, N-S\} \).

If \( x = 0 \) then \( g(\alpha, y, 0) = p_3S + p_3y\psi(\alpha)D_3 \), which is convex since \( \psi \) is convex.

If \( x = \hat{x} \) then in the excess capacity case \( g(\alpha, y, A(\alpha, y)) = p_2(1-\alpha)yD_2 + p_1((a+by)d_1 + \alpha yD_2) \) which is linear in \( \alpha \). In the scarce capacity case, if bumping is not allowed then \( g \) is piecewise linear convex. By our assumption, if \( \alpha \leq \hat{\alpha} \) then \( \psi(\alpha) = 0 \) and so \( \frac{\partial g}{\partial \alpha} = -yD_2(p_2 - p_1) \). If \( \alpha > \hat{\alpha} \) then \( \psi' = D_2/D_3 \) and \( \frac{\partial g}{\partial \alpha} = -yD_2(p_2 - p_1E(\alpha, y)) + yD_2p_3(1 - E(\alpha, y)) \), where \( E(\alpha, y) = \frac{N-S-\psi(\alpha)yD_3}{A(\alpha, y) - \psi(\alpha)yD_3} < 1 \) and \( \frac{\partial E}{\partial \alpha} = -\frac{y((a+by)d_1 + yD_2 - N)(D_2 - \psi(D_3))}{(A(\alpha, y) - \psi(\alpha)yD_3)^2} = 0 \). The function is convex because \( \frac{\partial E}{\partial \alpha} \bigg|_{\alpha = \hat{\alpha}} > 0 \). If bumping is allowed, then

\[
\frac{\partial^2 g}{\partial \alpha^2} = \left( yD_2((a+by)d_1 + yD_2 - N)(p_3 - p_1) \right) \left( A(\alpha, y) - \psi(\alpha)D_3 \right)^3 \]

(17)

because the numerator in the first term in (17) is positive since in the scarce capacity case \((a+by)d_1 + yD_2 \geq N\), and \( p_3 > p_1 \) by the definition. The denominator is positive since \( A(\alpha, y) - \psi(\alpha)D_3 = (a+by)d_1 + \alpha yD_2 - \psi(\alpha)D_3 \geq (a+by)d_1 + yD_2 \geq 0 \) as \( \alpha D_2 \geq \psi(\alpha)D_3 \) (by the definition of \( \psi \), since the number of waiting customers of class-2 is non-negative). In the second term, \( A(\alpha, y) - \psi(\alpha)D_3 \geq 0 \) since \( \psi \) is convex and by the above \( A(\alpha, y) - \psi(\alpha)D_3 \geq 0 \). And \( 2y(\psi(D_3 - D_2)(\psi(D_3 - D_2)A(\alpha, y) - y\psi(D_2)) \geq 0 \) because by the assumptions on \( \psi \) both elements of the product are negative.

If \( x = N-S \) then in the case of the excess capacity \( N-S > A \) and therefore \( g(\alpha, y, N-S) = p_2(1-\alpha)yD_2 + p_1((a+by)d_1 + \alpha yD_2) \), which is linear in \( \alpha \). In the case of scarce capacity \( N-S > \hat{x} \) and therefore \( g(\alpha, y, N-S) = p_2(1-\alpha)yD_2 + p_1(N - (1-\alpha)yD_2) + (p_3 - p_C)\psi(\alpha)yD_3 - (N - (1-\alpha)yD_2)/A(\alpha, y) \). For this,
\[
\frac{\partial^2 g}{\partial \alpha^2} = \left(\frac{yD_3((a+by)d_1+yD_2-N)(p_3-p_C)}{A(\alpha,y)^3} \right) \left( A(\alpha,y)^2 \psi''_\alpha + 2yD_2(y\psi D_2 - \psi'_\alpha A(\alpha,y)) \right) \geq 0
\]

because the first term in (18) is positive since in the scarce capacity case \((a+by)d_1+yD_2 \geq N\), and \(p_3 > p_C\) by the definition. And in the second term, \(\psi\) is convex, and by the assumption \(y\psi D_2 \geq \psi'_\alpha A(\alpha,y)\). Therefore \(g(\alpha,y,x)\) is convex in \(\alpha\) for \(x \in \Pi_t\), and the result follows.

Lastly note that in the no-bumping case \(p_C = p_3\) and so the above derivative is zero. Therefore \(g\) is a linear decreasing function of \(\alpha\), and since \(h\) increases in \(x\), future revenue is therefore decreasing in \(x\). Single period revenue decreases in \(x\) for \(x > \hat{x}\) and therefore the total revenue also decreases on \(x > \hat{x}\). Hence \(x^* \in \{0, \hat{x}\}\). Q.E.D.

**Appendix B: Simplified Models with Non-perfectly Correlated Demands**

In our simplified models we assumed that the demands between classes -1 and -2 are either independent, or perfectly positively or negatively correlated. That allowed us to present the solution in closed form. By observing initial sales, and knowing the waiting parameter, we could estimate the exact number of waiting customers.

In this section we extend the simplified models to the cases of non-perfectly correlated demands. We show that (subject to a certain regularity condition) it is sufficient to know the upper (lower) bound on the class-1 demand to be able to determine the optimal number of units on sale in the closed form in the case of positive (negative) correlation. We refer to the simplified model with two classes of Sections 5.2 and 5.3 as the *main* model.

To model non-perfectly correlated demands we assume that whenever the class-2 demand multiplier is \(y_t\), the class-1 demand multiplier can be either the same \(y_t\), with probability \(q\), or \(\epsilon y_t\), \(\epsilon > 0\) with probability \(1 - q\). Such a treatment is an extension of the high/low demand models that are frequent in the dynamic pricing literature (e.g., Su 2007). Recall that whenever the class-2 demand multiplier is \(y\), the class-1 demand multiplier is \(a+by\). For the ease of exposition in this section we assume \(a = 0\) and \(b = 1\); it is not difficult (but messy) to verify that the results hold for general \(a, b\). Let \(A(\alpha,y) = \alpha y d_2 + y d_1\), and let \(A_t(\alpha,y) = \alpha y d_2 + \epsilon y d_1\). Redefine similarly from (10) \(\hat{x}_t\) and \(\hat{x}^*_t\). Suppose \(\epsilon > 1\); observe \(A(\alpha,y) < A_t(\alpha,y)\) and so \(\hat{x}_t < \hat{x}^*_t\).

From (9) denoting the expected single period revenue as \(g_t\) we obtain:

\[
g_2(\alpha_t, y_t, x_t) = p_2S_t + p_1 \left( q \min[x_t, A_t(\alpha_t, y_t)] + (1-q) \min[x_t, A_{\epsilon t}(\alpha_t, y_t)] \right) + p_2\alpha_t y_t D_2 \left( q(1 - \frac{\min[x_t, A_t(\alpha_t, y_t)]}{A_t(\alpha_t, y_t)}) + (1-q)(1 - \frac{\min[x_t, A_{\epsilon t}(\alpha_t, y_t)]}{A_{\epsilon t}(\alpha_t, y_t)}) \right) - qp_C \left( \alpha_t y_t D_2 \left( \frac{1 - \min[x_t, A_t(\alpha_t, y_t)]}{A_t(\alpha_t, y_t)} \right) - (N-S_t - x_t) \right)
\]
-\left(1 - q\right) p_c \left( \alpha_t y_t D_2 \left( 1 - \frac{\min\left[x_t, A_t(\alpha_t, y_t)\right]}{A_t(\alpha_t, y_t)} \right) - \left( N - S_t - x_t \right) \right)^+ \quad (19)

Note that this revenue function resembles the main model, (9), if \( \epsilon = 1 \) or, equivalently, \( q = 1 \).

In the main model we show that in each period there are only two possibly optimal solutions: \( x_t = 0 \) or \( x_t = \hat{x}_t \). The result holds because the revenue function in the single period is piece-wise linear with a breakpoint at \( \hat{x}_t \), non-increasing for \( x > \hat{x}_t \), and the future revenue is decreasing and convex. Thus the total revenue was a piece-wise convex function, which is decreasing for \( x > \hat{x}_t \), and so the optimal solution is either \( x_t = 0 \) or \( x_t = \hat{x}_t \).

Below we show that in this new model with non-perfectly correlated demands an equivalent result holds, but rather for \( x_t = 0 \) or \( x_t = \hat{x}_{ct} \). The logic is also similar – we show that the single-period revenue is piece-wise convex on \( x \in [0, \hat{x}_{ct}] \) and non-increasing for \( x > \hat{x}_{ct} \). That ensures that in the single period the optimal solution is either \( x_t = 0 \) or \( x_t = \hat{x}_{ct} \). Convexity of the future revenue then follows by the same argument as in the main model upon noting that \( \hat{x}_{ct} \) in the new model is equal to \( \hat{x}_t \) of the old model but with nominal demand \( d_1 \) redefined as \( \epsilon d_1 \).

It therefore remains to show piece-wise convexity and decreasing property for the single-period revenue function \( g^2 \). As in the main model we proceed by considering the excess and scarce capacity cases. In the new model these capacity scenarios are transformed into three cases:

Case 1: Always excess capacity, i.e., \( A_t(\alpha, y) + S \leq N \). In this case bumping does not occur and so the revenue function simplifies to:

\[
g^2_t(\alpha_t, y_t, x_t) = p_2 S_t + p_1 (q \min[x_t, A_t(\alpha_t, y_t)] + (1 - q) \min[x_t, A_t(\alpha_t, y_t)]) + p_2 \alpha_t y_t D_2 \left( q (1 - \frac{\min[x_t, A_t(\alpha_t, y_t)]}{A_t(\alpha_t, y_t)}) + (1 - q) (1 - \frac{\min[x_t, A_t(\alpha_t, y_t)]}{A_t(\alpha_t, y_t)}) \right) \quad (20)
\]

If \( x \leq \hat{x}_t \) then \( g^2 \) is linear in \( x \) with slope \( p_1 - p_2 \alpha y D_2 \frac{q A_t(\alpha, y) + (1 - q) A_t(\alpha, y)}{A(\alpha, y) A_t(\alpha, y)} \).

If \( \hat{x}_t \leq x \leq \hat{x}_{ct} \) then \( g^2 \) is linear in \( x \) with slope \( (1 - q) \left( p_1 - \frac{p_2 \alpha y D_2}{A_t(\alpha, y)} \right) \).

If \( x \geq \hat{x}_{ct} \) then same as in the main model, \( g^2 \) is independent of \( x \).

By taking the difference of the first and second slopes, it is not difficult to check that \( g^2 \) is convex on \( x \in [0, \hat{x}_{ct}] \) if \( p_1 - \frac{p_2 \alpha D_2}{\alpha D_2 + d_1} \leq 0 \). The latter condition is the familiar “bang-bang” threshold from the main model (see the proof of Theorem 3 for \( a = 0, b = 1 \)). The inequality holds if the difference in prices, \( p_2 - p_1 \) is high enough so that the fraction of waiting customers \( \alpha \) is never below a certain point that makes the threshold equal zero. If that is the case, i.e., when the discounted price is low enough, then the revenue function is convex.

Case 2: Probabilistic scarce capacity, i.e., \( A(\alpha, y) + S \leq N \leq A_t(\alpha, y) + S \). In this case bumping
can occur only if the class-1 demand is high (has multiplier $\epsilon y$, which happens with probability $1-q$). In that case the revenue function is:

$$g_2(t)(\alpha_t, y_t, x_t) = p_2 S_t + p_1 \left( q \min[x_t, A_t(\alpha_t, y_t)] + (1-q) \min[x_t, A_{ct}(\alpha_t, y_t)] \right)$$

$$+ p_2 \alpha_t y_t D_2 \left( q \left( 1 - \frac{\min[x_t, A_t(\alpha_t, y_t)]}{A_t(\alpha_t, y_t)} \right) + (1-q) \left( 1 - \frac{\min[x_t, A_{ct}(\alpha_t, y_t)]}{A_{ct}(\alpha_t, y_t)} \right) \right)$$

$$- (1-q) p_C \left( \alpha_t y_t D_2 \left( 1 - \frac{\min[x_t, A_{ct}(\alpha_t, y_t)]}{A_{ct}(\alpha_t, y_t)} \right) - (N - S_t - x_t) \right)$$

(21)

For this observe that depending on the values of $x$, there are the same three ranges as in the Case 1 above. In the third range, $g_2$ is decreasing in $x$ same as in the main model, and in either of the first two the slope is modified by deducting the same penalty-related term (the multiple of $p_C$). Thus the difference between slopes is the same as in case 1, and so convexity on the $x \in [0, \hat{x}_{ct}]$ range follows by the same argument.

Case 3: Always scarce capacity, i.e., $A(\alpha, y) + S \geq N$. In this case the revenue function is given by (19) and depending on the value of $x$ the same three ranges as in Case 1 can occur. In the third range the slope in $x$ is negative by the same argument as in the main model. The slope in the first range is same as in Case 1 with two penalty terms deducted, and the slope in the second range has only the second penalty term deducted. Thus the slope over the second range is larger; convexity over $x \in [0, \hat{x}_{ct}]$ follows.

In sum, the single-period revenue function is convex on $x \in [0, \hat{x}_{ct}]$ and non-increasing on $x > \hat{x}_{ct}$, and so as we discussed above the results of Theorems 3 and 4 hold with $x^* \in \{0; \hat{x}_{ct}\} = \Pi_{ct}$. The sufficient condition for this result is that $p_1 - \frac{p_2 D_2}{\alpha D_2 + d_1} \leq 0$. This inequality holds if the difference in prices, $p_2 - p_1$ is high enough so that the fraction of waiting customers $\alpha$ is never below a certain point that makes the threshold equal zero.

Managerially, it implies that if the discounted price is low enough, then the firm does not need to know the exact number of discount-only class-1 customers. Rather, it is sufficient to know the lower bound on the number of discount customers.

Further, it is not hard to verify that if $\epsilon < 1$ then the condition for convexity becomes $\alpha \geq \bar{\alpha}$, and so the logic flips – the discounted price has to be high enough so that the number of waiting customers is always low enough, and then it is sufficient for the firm to only know the upper bound on the number of discount-only class-1 customers in the market.

**Appendix C: Random Allocation of Discounted Units**

We tested whether it is necessary in our model to allocate inventory in a random first come, first serve (FCFS) manner or if we are justified in assuming a proportional allocation. Under FCFS,
assuming all waiting customers arrive according to the same process, the allocation would be determined by the realization of a Hypergeometric random variable, implying the allocation function is given by $B_{FCFS} = \psi(\alpha)yD_3 - \text{Hypergeometric}(x, \psi(\alpha)yD_3, A)$. Under our proportional allocation assumption, $B = \psi(\alpha)yD_3(1 - \min\left\{x, \frac{A}{A} \right\}) = E[B_{FCFS}]$, thus proportional allocation substitutes $B_{FCFS}$ with its expected value.

We compared the analytical optimal policy derived for proportional allocation with the policy optimal for random allocation derived using simulation. Using the scenarios presented in Section 7, we observed that the simulated and optimal policies in most cases put the same number of units on sale. When non-zero, the optimal number of units to place on sale is typically large (e.g., in Figure 2 (a) $\hat{x} = 75...90$ units). Therefore, by the central limit theorem, the relative error of using the expectation as opposed to the realization of $B_{FCFS}$ is small. As a result, the revenue generated under the optimal policy when the discounted units are allocated in a random, FCFS manner is also close to that of the case when they are proportionally allocated. In our simulations the difference between these revenues never exceeded 3 percent. Thus we conclude that for the examples we studied, the proportional allocation assumption is justified.