Risk Management for Derivatives

“This Greeks are coming the Greeks are coming!“

Managing risk is important to a large number of individuals and institutions. The most fundamental aspect of business is a process where we invest, take on risk and in exchange earn a compensatory return. The key to success in this process is to manage your risk-return trade-off. Managing risk is a nice concept but the difficulty is often measuring risk. There is a saying “what gets measured gets managed.” To alter this slightly, “What cannot be measured cannot be managed”. Hence risk management always requires some measure of risk. Risk in the most general context refers to how much the price of a security changes for a given change in some factor.

In the context of Equities, Beta\(^1\) is a frequently used measure of risk. Beta measures the relative risk of an asset. High Beta stocks or portfolios have more variable returns relative to the overall market than low Beta assets. If a Beta of 1.00 means the asset has the same risk characteristics as the market, then a portfolio with a Beta greater than one will be more volatile than the market portfolio and consequently is more risky with higher expected returns. Conversely assets with a Beta less than 1.00 are less risky than average and have lower expected returns. Portfolio managers use Beta to measure their risk-return trade-off. If they are willing to take on more risk (and return), they increase the Beta of their portfolio and if they are looking for lower risk they adjust the Beta of their portfolio accordingly. In a CAPM framework, Beta or market risk is the only relevant risk for portfolios.

For Bonds, the most important source of risk is changes in interest rates. Interest rate changes directly affect bond prices. Modified Duration\(^2\) is the most frequently used measure of how bond prices change relative to a change in interest rates. Relatively higher Modified Duration means more price volatility for a given change in interest rates. For both Bonds and Equities, risk can be distilled down to a single risk factor. For Bonds it is Modified Duration and for equities it is Beta. In each case, the risk can be measured and adjusted or managed to suit ones risk tolerance.

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\(^{1}\) For a discussion of Beta please see,  
In each of these cases, financial theory provides a measure of risk. Using these risk measures, holders of either bonds or equities can adjust or manage the risk level of the securities that they hold. What is nice about these asset categories is that they have a single measure of risk. Derivative securities are more challenging.

**Risk Measures for Derivatives**

In the discussion which follows we will define risk as the sensitivity of price to changes in factors that affect an asset’s value. More price sensitivity will be interpreted as more risk.

**The basic option pricing model**

A simple European Call option can be valued using the Black-Scholes Model\(^3\):

\[
\text{European Call Option Value} = UAV \cdot N(d_1) - X \cdot e^{-r_f T} \cdot N(d_2)
\]

Where

\[
N(\cdot) = \text{Cumulative Standard Normal Function},
\]

\[
d_1 = \frac{\ln \left( \frac{UAV}{X} \right) + \left( r_f + \frac{1}{2} \cdot \sigma^2 \right) \cdot T}{\sigma \cdot \sqrt{T}},
\]

\[
d_2 = d_1 - \sigma \cdot \sqrt{T},
\]

\[
UAV = \text{Underlying Asset Value},
\]

\[
T = \text{Time to maturity},
\]

\[
\sigma = \text{UAV volatility}
\]

\[
X = \text{Exercise (Strike) Price}
\]

\[
r_f = \text{Risk-free rate}
\]

This very basic option pricing model demonstrates that the value of a European call option depends on the value of five factors: Underlying Asset Value, Risk-free rate, volatility, Time to maturity, and Exercise price. If the value of any one of these factors should change, then the value of the option would change. Of the five inputs or factors, one is fixed (the exercise price), another is deterministic (time to maturity), and the other three change randomly over time (underlying asset value, volatility and the risk-free rate). Hence the risk associated with holding a European Call option is that the underlying asset value could change, the volatility could change, or the risk-free rate could change. Any change in the value of these things would change the value of the call option.

Our original definition of risk was; How much does the value of a call option change given a change in the value of the underlying factors? Based on this there need to be different risk measures for each factor. These risk measures are the “Greeks”. The include Delta, Gamma, Vega\(^4\) and Rho. The next sections will deal with each of these. The process is to examine how the Black-Scholes model value changes in response to a change in one of the inputs shown above.

**Delta**

Delta is a measure of how much the value of an option, forward or a futures contract values will change over a very short interval of time for a given change in the asset price. The simplest is the Delta for a long position in one share of stock. Since a $1 change results in a $1 change in value, the Delta is 1. A one dollar change in the stock results in a one dollar change in the value of the long position. Once we know how much the value of a position in stock, options or futures contracts will change for a given change in the price of the underlying asset (Delta), we can then use this information to hedge the underlying price risk. This hedging of price risk is often referred to as “Delta Hedging”.

**Calculating Delta for Call Options and Put Options**

The most obvious factor that affects an option’s value is the value of the underlying asset. Using the basic Black-Scholes model, we can take the partial derivative with respect to UAV.

\[
\frac{\Delta \text{Call}}{\Delta \text{UAV}} = \frac{\partial \text{Call}}{\partial \text{UAV}} = \frac{\partial (UAV \cdot N(d_1) - X \cdot e^{-rT} \cdot N(d_2))}{\partial \text{UAV}}.
\]

It turns out that the solution\(^5\) to this is

\[
\frac{\partial \text{Call}}{\partial \text{UAV}} = N(d_1),
\]

where

\[
d_1 = \frac{\ln \left( \frac{UAV}{X} \right) + \left( r_f + \frac{1}{2} \cdot \sigma^2 \right) \cdot T}{\sigma \cdot \sqrt{T}}.
\]

This is a general result. It is the relationship between the value of the call option and the value of UAV.

\(^4\) Vega is not actually a “Greek” letter but we can be a bit generous with our terminology.

\(^5\) It is important to note that taking the partial derivative of the full Black-Scholes model is more complicated than it appears when you look at the solution. You can see this by noting that the value of \(d_1\) also includes the value UAV. However, conveniently everything cancels out and you get the rather simple expression shown.
In practice, most people want to know the relationship between the call value and the stock price and this is what is usually called the \textbf{Delta} of a call option.

\[
Delta = \partial \text{Call} / \partial \text{UAV} \cdot \partial \text{UAV} / \partial S = \partial \text{Call} / \partial S
\]

It is important to note that the value of Delta for a particular call will depend on how UAV is defined. The simplest case is for call options on non-dividend paying stocks. In this case, UAV = stock price. As such, \( \partial \text{UAV} / \partial S = 1 \) and

\[
Delta = \partial \text{Call} / \partial S = \partial \text{Call} / \partial \text{UAV} \cdot \partial \text{UAV} / \partial S = N(d_1) \cdot \partial \text{UAV} / \partial S = N(d_1) \cdot 1 = N(d_1).
\]

Consider a European Call option on stock XYZ with a strike price of $50 and a time to maturity of 90 days. The XYZ’s stock price today is $55, the estimated volatility is .35 and the current 90 day risk-free rate is 2.5%. Given these parameters, the Black-Scholes value is

\[
\text{Call} = \$6.877 = \begin{cases} 
\text{UAV} &= \$55 \\
\text{Maturity} &= 90 \text{days} \\
X &= \$50 \\
R_f &= 2.5\% \\
\sigma &= .35 
\end{cases}
\]

and the Delta would be

\[
Delta = N \left( \frac{\ln \left( \frac{S}{X} \right) + \left( R_f + \frac{1}{2} \cdot \sigma^2 \right) \cdot T}{\sigma \cdot \sqrt{T}} \right) = N \left( \frac{\ln \left( \frac{55}{50} \right) + \left[ \frac{.025 + \frac{1}{2} \cdot (.35)^2}{.35} \right] \cdot \frac{90}{365}}{\frac{.35}{\sqrt{90/365}}} \right) = .749.
\]

The Call price is $6.877 and Delta of this call option is .749.

Given a Delta of .749, we would expect that a $0.10 change in the stock price will result in an increase in the value of the call of $0.0749. Changing the UAV from $55 to $55.10 in the Black-Scholes model above gives a call option price of $6.952 and an actual change of $0.0749. If the stock price were to increase to $56, then the predicted price change given the Delta would be $0.749 and the actual Black-Scholes calculated price change would be $0.765. They are not exactly the same and this indicates that Delta is really only accurately measures the option price change for small changes in the underlying stock price.

In this case, \( Delta = N(d_1) \), where \( N(d_1) \) is the cumulative standard normal distribution. Given this definition\(^6\) of Delta, it has a minimum of 0 and a maximum of 1.00.

\(^6\) Since it is a cumulative distribution, it can only take on values between 0 and one.
options that are deep out of the money will have a Delta of approximately 0. This reflects that fact that if the option is way out of the money, its value is very small and small changes in the stock price will not materially impact on the value of the call option. Hence a small change in stock price will not affect the call’s value and the Delta would be close to zero. From the example above, if the stock price were $30 and not $55, the call value would be $0.004 and the Delta = 0.002. The low option value reflects the very small probability that the stock price will rise above $50 in the next 90 days. The small Delta reflects that fact that even if the stock price goes to $31 that is not materially going to increase the probability of the price rising above $50 and as such, there will be very little change in the value of the call option.

On the hand, for call options which are deep in the money, the Delta would be 1.00. If the stock price were $70, the call value would be $20.401 and the Delta = .980. Here the Delta is almost 1.00 because a $1.00 change in price gets incorporated into the intrinsic value and there is only a very small negative change in the time value. Exhibit 1 shows the call option value for this option given different stock prices. Here you get the familiar result that the relationship between stock prices and call prices are flat at low stock prices and linear for high prices. In between, they are convex. Delta starts at zero and increases as the stock price increases. The maximum value of Delta is 1.00.

Exhibit 1.
European Call Option (X=$50, $0.35, T= 90 days, $r_f = 2.5%)

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7 For call options, when the stock price is below the exercise price, the time value reflects the probability that the stock price will be above the exercise price at maturity. As such, as the stock price gets closer to the exercise price the time value increases. When the stock price is above the exercise price, the time value reflects the probability that the stock price will be below the exercise price. The value of an option over owning the stock is the fact that at maturity you do not have to buy it if the stock price is below the exercise price. Hence the higher the stock price relative to the exercise price the less the valuable the option is relative to the stock. That is the time value decreases as the stock price increases. Consequently a $1 change in stock price increases the intrinsic value by $1 but the time value actually decreases and the net change in value of the call option is less than $1.
The Call value can be divided into the intrinsic value and a time value. Exhibit 2 shows the intrinsic value and time values based on the call option values in Exhibit 1. Beginning at a price below the exercise price the call option value is all time value. As the stock price increases the time value does not increase at a rate of one for one. In fact on the far left of the exhibit the time value is essentially flat and Delta = 0. Time value increases at an increasing rate as we move toward the exercise price. In fact, time value is greatest when the stock price is equal to the exercise price. As the stock price increases beyond the exercise price, the intrinsic value increases on a one for one basis but the time value actually begins to decrease. This results in a Delta less than 1.00 until we get to the far right of the exhibit where the time value line goes to zero and is again essentially flat. Here the Delta is 1.00.

Exhibit 2.
European Call Option (X=$50, σ = .35, T= 90 days, r_f = 2.50%)

In order to hedge price risk, we estimate the Delta of the security and then take a position in another asset with a negative Delta Value. For example if the Stock price were $55 and we were the holder (buyer) of the call option above, then Delta=.749. To hedge this position we could use would need to find another call option on the same stock. For example, another call option, with the same maturity but an exercise price of $45, would have a price of $10.769 and a Delta of .899. If we write (sell) .833 of this call option, the Delta would be -.749 (.833 x .899 = .749). Combining the long position in the call and the short position in the second call, results in a net Delta of 0.

Call option Delta for dividend paying stocks

The Delta for a dividend paying stock presents a slightly different challenge. For dividend paying stocks, the UAV does not equal the stock price. In the known dividend approach to valuing call option, the Underlying Asset Value is UVA = S – (Present Value of Dividends). In this formulation, since the present value of the dividends does not depend on S, $\frac{\partial UAV}{\partial S} \neq 1$ and
\[
\Delta = \frac{\partial \text{Call}}{\partial S} = \frac{\partial \text{Call}}{\partial UAV} \cdot \frac{\partial UAV}{\partial S} = N(d_1) \cdot \frac{\partial UAV}{\partial S},
\]
\[
\Delta = N(d_1) \cdot 1
\]
\[
\Delta = N(d_1)
\]

where
\[
d_1 = \frac{\ln\left(\frac{UAV}{X}\right) + \left(r_f + \frac{1}{2} \cdot \sigma^2\right) \cdot T}{\sigma \cdot \sqrt{T}} = \frac{\ln\left(\frac{S - PVofDividends}{X}\right) + \left(r_f + \frac{1}{2} \cdot \sigma^2\right) \cdot T}{\sigma \cdot \sqrt{T}}.
\]

However, if we used the dividend yield approach, then UAV = \(S \cdot e^{-d_y \cdot T}\), where \(d_y\) is the dividend yield and here the adjustment for the dividend does depend on \(S\). As such, \(\frac{\partial UAV}{\partial S} = e^{-d_y \cdot T}\). Hence for call value based on using the dividend yield model,
\[
\Delta = \frac{\partial \text{Call}}{\partial UAV} \cdot N(d_1) \cdot e^{-d_y \cdot T}
\]

where,
\[
d_1 = \frac{\ln\left(\frac{UAV}{X}\right) + \left(r_f + \frac{1}{2} \cdot \sigma^2\right) \cdot T}{\sigma \cdot \sqrt{T}} = \frac{\ln\left(\frac{S \cdot e^{-d_y \cdot T}}{X}\right) + \left(r_f + \frac{1}{2} \cdot \sigma^2\right) \cdot T}{\sigma \cdot \sqrt{T}}.
\]

**European put options Delta**

For European Put Options we can use put-call parity to estimate Delta. The Put-Call parity no arbitrage relationship for European options on non-dividend paying stocks is, Stock + Put = Call + Bond. If we take the partial derivative of the put-call parity relationship with respect to stock price we get
\[
\frac{\partial (\text{Stock})}{\partial S} + \frac{\partial \text{Put}}{\partial S} = \frac{\partial \text{Call}}{\partial S} + \frac{\partial (\text{Bond})}{\partial S} + \frac{\partial \text{Put}}{\partial S} = \frac{\partial \text{Call}}{\partial S} + \frac{\partial \text{Put}}{\partial S} = 0
\]
\[
\frac{\partial \text{Put}}{\partial S} = \frac{\partial \text{Call}}{\partial S} - 1
\]
\[
\frac{\partial \text{Put}}{\partial S} = N(d_1) - 1
\]
\[
\Delta_{\text{Put}} = \Delta_{\text{Call}} - 1
\]
Calculating Deltas for Forward and Futures Contracts

Consider a Futures contract for 1000 bushels of corn with contract price $K$ and delivery at time $T$. If at the end of the day the price of delivery at time $T$ is $F_T$, where $F_T = \text{Spot} \cdot e^{r\cdot T}$. Then because the contract is settled daily, the value of the Futures contract is

$$\text{Value of Futures Contract} = VF$$

$$VF = F_T - K = \text{Spot} \cdot e^{r\cdot T} - K.$$ 

The Delta of the value of a Futures Contract is

$$\text{Delta Futures} = \frac{\partial VF}{\partial \text{Spot}} = \frac{\partial (F_T - K)}{\partial S} = \frac{\partial (\text{Spot} \cdot e^{r\cdot T} - K)}{\partial S} = e^{r\cdot T}.$$ 

This tells us that for every $1$ change in the spot price the value of the futures contract changes by $e^{r\cdot T}$. Hence if we wished to hedge the spot price of 1000 bushels of corn overnight using this particular Futures contract we would use futures contract for a total of $1000 \cdot e^{r\cdot T}$ bushels.

Forward Contracts Delta

Here consider you have the Buy-side of a Forward contract for 1000 bushels of corn with contract price $K$ and delivery at time $T$. If at the end of the day the forward price$^8$ of delivery at time $T$ is $F_T$, where $F_T = \text{Spot} \cdot e^{r\cdot T} > K$, then to realize the value of the contract you would have to take the sell side of a contract that matures at time $T$ with forward price $F_T$. Because these are forward contracts, the value of the payoff is

$$\text{Value of Forward Contract} = VFW$$

$$VFW = (F_T - K) \cdot e^{-r\cdot T} = (\text{Spot} \cdot e^{r\cdot T} - K) \cdot e^{-r\cdot T} = \text{Spot} - K \cdot e^{-r\cdot T}.$$ 

The Delta of the value of a Forward contract is just

$$\text{Delta Forward} = \frac{\Delta VFW}{\Delta \text{Spot}} = \frac{\partial VF}{\partial \text{Spot}} = \frac{\partial (F_T - K)}{\partial S} = \frac{\partial (S - K \cdot e^{-r\cdot T})}{\partial S} = 1.00.$$ 

This tells us that for every $1$ change in the spot price the value of the forward contract changes by $1.00$. Hence if we wished to hedge the spot price of 1000 bushels of corn overnight using this particular Futures contract we would use futures contract for a total of 1000 bushels.

$^8$ In this simple example we will ignore storage costs, spoilage and convenience yields.
GAMMA (Γ)

From Exhibit 1 we see that for options Delta changes as the stock price changes. If we set up a hedge using Delta, as the stock price changes, Delta changes and as such we need to adjust hedge ratio. How much Delta changes as the stock price changes is important. The rate of change in Delta is called Gamma.

Calculating Gamma for Call and Put options

To capture this, we calculate Gamma in the following way,

$$\Gamma = \Delta \text{Delta} \cdot \frac{\Delta UAV}{\Delta S} = \frac{\partial (\text{Delta})}{\partial d_t} \cdot \frac{\Delta d_t}{\partial UAV} \cdot \frac{\partial UAV}{\partial S} = N'(d_t) \cdot \frac{1}{UAV \cdot \sigma \cdot \sqrt{T}} \cdot \frac{\partial UAV}{\partial S}$$

Again this is a very general formulation that includes UAV. However, since Gamma like Delta is defined in terms of change in Delta relative to a change in stock price, the term $$\frac{\partial UAV}{\partial S}$$ is important. For non-dividend paying stocks, the UAV=S, $$\frac{\partial UAV}{\partial S}=1$$, and the function for Gamma reduces to

$$\Gamma = N'(d_t) \cdot \frac{1}{S \cdot \sigma \cdot \sqrt{T}}$$

where $$N'(d_t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_t^2}$$. A call option with an exercise price of $50, r_f=2.5\%$, T=90 days, $$\sigma = .35$$, and stock price of $55 has a Gamma = 0.0333. Exhibit 3 shows the relationship between Delta and Gamma as the stock price changes for this call option. The largest Gamma is when the stock price is just below the exercise price.

Exhibit 3.

High values of Gamma indicate that small changes in the stock price result in relatively large changes in Delta. As such, individuals or institutions doing delta hedging for
options where the current stock price is close to the exercise price have to adjust the hedge more frequently.

Because UAV for dividend paying stocks does not equal the stock price, Gamma for options on dividend paying stocks is slightly different. If we are using a known dividend model, then Gamma is

\[
\Gamma = \Delta\text{Delta}_{\text{UAV}}/\Delta\text{UAV}/\Delta S = \left(\partial\text{Delta}/\partial d_i\right)\left(\partial d_i/\partial \text{UAV}\right)\left(\partial \text{UAV}/\partial S\right) = N'(d_i) \cdot \frac{1}{\text{UAV} \cdot \sigma \cdot \sqrt{T}} \cdot \partial \text{UAV}/\partial S
\]

\[
\Gamma = N'(d_i) \cdot \frac{1}{\text{UAV} \cdot \sigma \cdot \sqrt{T}} \cdot \ln\left(\frac{\text{UAV}}{X}\right) + \left(r_f + \frac{1}{2} \cdot \sigma^2\right) \cdot T
\]

where \(d_i = \frac{\ln\left(\frac{S}{X}\right) + \left(r_f + \frac{1}{2} \cdot \sigma^2\right) \cdot T}{\sigma \cdot \sqrt{T}}\) and \(\text{UVA} = S - \text{Present Value of Dividends}\).

If we are using a dividend yield model, then Gamma is

\[
\Gamma = \Delta\text{Delta}_{\text{UAV}}/\Delta\text{UAV}/\Delta S = \left(\partial\text{Delta}/\partial d_i\right)\left(\partial d_i/\partial \text{UAV}\right)\left(\partial \text{UAV}/\partial S\right) = N'(d_i) \cdot e^{-d_i \cdot \tau} \cdot \frac{1}{\text{UAV} \cdot \sigma \cdot \sqrt{T}} \cdot \partial \text{UAV}/\partial S
\]

\[
\Gamma = N'(d_i) \cdot e^{-d_i \cdot \tau} \cdot \frac{1}{S \cdot \sigma \cdot \sqrt{T}} \cdot \ln\left(\frac{S}{X} \cdot e^{-d_i \cdot \tau}\right) + \left(r_f + \frac{1}{2} \cdot \sigma^2\right) \cdot T
\]

where \(d_i = \frac{\ln\left(\frac{S}{X} \cdot e^{-d_i \cdot \tau}\right) + \left(r_f + \frac{1}{2} \cdot \sigma^2\right) \cdot T}{\sigma \cdot \sqrt{T}}\).

Note\(^9\) that since the \(\Delta\text{Delta}_{\text{Put}} = \Delta\text{Delta}_{\text{Call}} - 1\), the Gamma of a Put option is equal to

\[
\frac{\partial \Delta\text{Delta}_{\text{Put}}}{\partial S} = \frac{\partial \Delta\text{Delta}_{\text{Call}}}{\partial S}
\]

and the Gamma of a Put option is equal to the Gamma of a call option with the same characteristics.

**Vega (\(v\))**

Vega is the relationship between the option value and changes in volatility. Again we use the Black-Scholes model and ask the question of all other things equal, how does the value of a call option change with a change in volatility? The answer is Vega

\[
\text{Vega} = v = \Delta\text{Call}/\Delta\sigma = \text{UAV} \cdot \sqrt{T} \cdot N'(d_i).
\]

\(^9\) Please see the section on Delta’s.
Again using the call option presented above with UAV=$55, T=90 \text{ days}, \sigma=.35, X=$50 and \( r_f = 2.50\% \), Vega = 8.701. This implies that a change in the volatility from .35 to .36 would result in a $0.0870 change in the call option price\(^{10}\).

Vega for European call options on non-dividend paying stocks where UAV = S is

\[
Vega = \nu = \frac{\Delta \text{Call}}{\Delta \sigma} = S \cdot \sqrt{T} \cdot N'(d_1)
\]

Vega for European call options on dividend paying stocks is either

Vega for known dividend model \(\nu = \frac{\Delta \text{Call}}{\Delta \sigma} = (S - PV\text{dividends}) \cdot \sqrt{T} \cdot N'(d_1),\)

or

Vega for dividend yield model \(\nu = \frac{\Delta \text{Call}}{\Delta \sigma} = (S \cdot e^{-d_1 T}) \cdot \sqrt{T} \cdot N'(d_1),\) where \(d_1\) is defined with the appropriate UAV value in each case.

Exhibit 4
European Call Option (X=$50, T = 90 \text{ days}, \sigma = .35, r_f = 2.50\%)

Rho
Rho\(^{11}\) measures the sensitivity of a call option price to changes in the risk-free rate. It is defined as

\[
Rho = \frac{\Delta \text{Call}}{\Delta r_f} = X \cdot T \cdot e^{-r_f T} \cdot N(d_2).
\]

\(^{10}\) The call option price would change from $6.877 (\(\sigma=.35\)) to $6.965 (\(\sigma=.36\)), for a change of $0.0877.

\(^{11}\) Note that I will use the roman letters for rho. The Greek letter \(\rho\) is very often used as the symbol for correlation coefficient. I have not used it here to avoid confusion.
The value of Rho for our example, UAV=$55, T= 90 days, X= $50, \sigma = .35 and r_f = 2.5\% is 8.459 and a change in the risk-free rate from 2.50\% to 2.60\% would result in a change of $0.008459 in the call option price.

Note that d_2 should include the definition of UAV consistent with the no dividend, the known dividend or the dividend yield models. The Rho for a Put is again derived based on Put-Call Parity,

$$\text{Stock} + \text{Put} = \text{Call} + \text{Bond}$$
$$S + \text{Put} = \text{Call} + X \cdot e^{-r_f \cdot T}$$
$$\text{Put} = \text{Call} + X \cdot e^{-r_f \cdot T} - S$$

and

$$Rho_{\text{Put}} = \frac{\partial P}{\partial r_f} = \frac{\partial C}{\partial r_f} + \frac{\partial (X \cdot e^{-r_f \cdot T})}{\partial r_f} + \frac{\partial S}{\partial r_f}$$

$$Rho_{\text{Put}} = \frac{\partial P}{\partial r_f} = \frac{\partial C}{\partial r_f} - X \cdot T \cdot e^{-r_f \cdot T} + 0$$

$$Rho_{\text{Put}} = X \cdot T \cdot e^{-r_f \cdot T} \cdot N'(d_2) - X \cdot T \cdot e^{-r_f \cdot T}$$

$$Rho_{\text{Put}} = -X \cdot T \cdot e^{-r_f \cdot T} \cdot (1 - N'(d_2))$$

$$Rho_{\text{Put}} = -X \cdot T \cdot e^{-r_f \cdot T} \cdot N(-d_2)$$

**Delta, Vega and Rho**

Delta, Vega and Rho represent measures of the three risk factors that one faces when you hold options either individually or in portfolios. Changes in the underlying asset value, volatility or the risk-free rate all affect option values.

**Theta (Θ)**

Theta measures the change in the value of an option with respect to time. This is fundamentally different than the other factors. Delta, Vega and Rho all measure price sensitivity to factors which can change randomly. Time on the other hand is deterministic. For example consider an option where the exercise price is $50, the current stock price is $55, risk-free rate is 2.5\%, time to maturity is 3 months and volatility is .35. One month from now the stock price, risk-free rate and/or volatility could be anything but with certainty the time to maturity will be two months and if all the other factors remained the same we would know with certainty what the change in the call option price would be. Clearly it would be less. This is known as the “time decay” of an option. Theta measures the rate of “time decay” and is equal to

$$Theta = \frac{\partial Call}{\partial T} = -\frac{UAV \cdot N'(d_1) \cdot \sigma}{2 \cdot \sqrt{T}} - r_f \cdot X \cdot e^{-r_f \cdot T} \cdot N(d_2)$$
For call options on non-dividend paying stocks, UAV = S. Once again using a call option with X=$50, T= 90 days, \( \sigma = .35 \), \( r_f = 2.5\% \) and current stock price of $55, the value of Theta would be

\[
\text{Theta} = - \frac{S \cdot N'(d_1) \cdot \sigma}{2 \cdot \sqrt{T}} - r_f \cdot X \cdot e^{-r_f \cdot T} \cdot N(d_2) = - \frac{55 \cdot N'(d_1) \cdot .35}{2 \cdot \sqrt{\frac{90}{365}}} - .025 \cdot 50 \cdot e^{-0.025 \cdot \frac{90}{365} \cdot N(d_2)}
\]

Theta = -7.033

where

\[
d_1 = \frac{\ln \left( \frac{S}{X} \right) + \left( r_f + \frac{1}{2} \cdot \sigma^2 \right) \cdot T}{\sigma \cdot \sqrt{T}} = \frac{\ln \left( \frac{55}{50} \right) + \left( .025 + \frac{1}{2} \cdot .35^2 \right) \cdot \left( \frac{90}{365} \right)}{.35 \cdot \sqrt{\frac{90}{365}}} = 0.6708
\]

and

\[
d_2 = d_1 - \sigma \cdot \sqrt{T} = 0.6708 - .35 \cdot \sqrt{\frac{90}{365}} = 0.4970.
\]

Exhibit 5 shows how Theta changes as stock price changes.

Exhibit 5.
European Call Option (X=$50, T = 90 days, \( \sigma = .35 \), \( r_f = 2.50\% \))

As we can see Theta or “time decay” is largest when the option is at “at the money”.

It is difficult to interpret Theta because often the time units are not clear. In the example about Theta = -7.033. This is expressed in terms of years. For example, a decrease in maturity from 90 days to 85 days is 5 days or 0.0137 years. With a Theta of -7.033, this decrease in maturity should decrease the option price by $0.0963 (-7.033 x .0137 =-
0.0963). It is very typical to express Theta in terms of calendar days, -0.0913 (-7.033/365) or in terms of trading days, -.0279 (-7.033/252). Bottom line is that you need to be clear what the time element is for defining Theta.

**Summary**

In this note we examined five different measures of how an option price can change given a change in one of the basic valuation parameters; Delta, Gamma, Vega, Theta and Rho.