Asset Pricing When Returns Are Nonnormal:
Fama-French Factors vs. Higher-order Systematic Co-Moments*

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Abstract

A growing literature contends that, since returns are not normal, higher-order co-moments matter to risk-averse investors. Fama and French (1993, 1995) find that non-market risk factors based on size and book-to-market ratio are priced by investors. We test the hypothesis that the Fama-French factors simply proxy for the pricing of higher-order co-moments. Using portfolio returns over various time horizons, we show that adding a set of systematic co-moments (but not standard moments) of order 3 through 10 reduces the explanatory power of the Fama-French factors to insignificance in almost every case.

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I. Introduction

According to the Capital Asset Pricing Model (CAPM), investors only price market risk. However, a growing literature identifies many non-market risk factors that appear to be priced. In particular, Fama and French (1993, 1995) find that the non-market risk factors SMB (the return on a portfolio of small stocks less the return on a portfolio of large stocks) and HML (the return on a portfolio of high book-to-market-value stocks less the return on a portfolio of low book-to-market-value stocks) are statistically important in explaining the cross-section of equity returns. There is, however, substantial debate regarding the economic meaning of SMB and HML. Fama and French (1993, 1996) suggest that book-to-market and size are proxies for firm distress. Lakonishok, Shleifer, and Vishny (1994) argue that book-to-market proxies for investor bias in earnings-growth extrapolation. Daniel and Titman (1997) find that SMB and HML pick up the co-movements of stocks with similar characteristics, so it is the characteristics, not the co-movements, that explain cross-sectional return variation. Rolph (2003) and Ferguson and Shockley (2003) argue that the Fama-French factors proxy for leverage effects. Berk (1995), Kothari, Shanken, and Sloan (1995), and Ferson, Sarkissian, and Simin (1999) argue that the explanatory power of SMB and HML are spurious. In this paper, we propose an alternative explanation.

If the CAPM holds, only the second-order systematic co-moment (beta) should be priced. Although the CAPM can be derived under various sets of assumptions, we argue that normality of returns is the crucial assumption: Without normality, the CAPM is unlikely to hold. Returns are not in general normal (see, e.g., Mandelbrot (1963), Ane and Geman (2000), and Aparicio and Estrada (2001)), and tend toward lognormality for longer intervals. We examine returns for daily, weekly, monthly, quarterly, and semi-annual intervals. We find evidence that normality is rejected for all five cases.

In addition, we find that factors besides the CAPM beta explain the cross-section of returns. The Fama-French (F-F) factors SMB and HML jointly provide statistically significant explanatory power across almost all the sample return horizons. We argue that SMB and HML proxy for measures of market risk not captured by the CAPM.

In the CAPM, investors care only about two moments – mean and variance – for portfolio returns and one co-moment – covariance – for security returns. In general, however, investors may care about higher moments – skewness, kurtosis, and so on – and higher co-moments – co-skewness, co-kurtosis, and so on. Scott and Horvath (1980) show that investors should have a negative preference for even moments and a positive preference for odd ones. Rubinstein (1973) derives an equation for the expected return in terms of an arbitrary number of co-moments. We test whether SMB and HML proxy for these co-moments. Others have looked at co-skewness, co-kurtosis, or both to explain returns (see, e.g., Bonsal and Viswanathan (1993), Dittmar (2002), Friend and Westerfield (1980), Harvey and Siddique (2000), Hung et al. (2003), Kan and Wang (2001), Kraus and Litzenberger (1976), Lim (1989), Perez-Quiros and Timmerman (2000), and Sears and Wei (1985)). We argue that there is no reason to stop with the fourth moment. Risk-averse investors are presumably very concerned about the risk of ruin. Furthermore, the popularity of lotteries, sweepstakes, out-of-the-money options, etc. implies that investors are also concerned about the right tail of the distribution. Variance, skewness, and kurtosis tell us something about the tails of the distribution, but they fall far short of specifying the tail precisely.

To see the importance of higher moments, consider two distributions: (1) a standard normal and (2) a simple mixture of distributions with a probability, p, ($0 < p < 1$) of drawing from a bilateral exponential with density $\frac{1}{2}\alpha e^{-\alpha|x|}$ and a probability $1-p$ of drawing from a standard normal. Since both distributions are symmetric, the odd moments are all zero for both. For appropriate choices of α and p we should be able to match, at least approximately, the variance and kurtosis for both. Thus, we can match the first five moments, yet the tails of the two distributions are completely different: The normal falls off much more quickly than the bilateral exponential. This example shows that knowing the first few moments does not imply that the shape of the tails is known.

Higher-order moments have been criticized for being unreliable and lacking intuition. We believe that both criticisms can be answered by looking at several co-moments. Each co-moment may individually be unreliable, but the set of co-moments, taken together, should not be. The idiosyncratic variations in individual co-moment estimates can be largely diversified away by looking at several co-moments. We have to expect multicollinearity, but our primary concern is to measure the effect of the entire set of higher co-moments. Similarly, whereas the intuition for an individual co-moment may not be obvious, the intuition for the entire set is clear: The set of co-moments is a measure of the likelihood of extreme outcomes, a matter of great importance to risk-averse investors. Using a set of co-moments to estimate the tails is rather like using a set of lagged values of inflation to estimate future inflation: In both cases we are not particularly interested in the individual members of the set but in the set as a whole. There are other statistical estimators we might use instead of co-moments, such as cumulants and the Hill estimator, but it is not clear to us that there is any advantage in using these alternatives instead of co-moments.

When returns are normal, the mean and variance suffice to describe the distribution completely. However, in general, to specify the tails perfectly requires an infinite number of moments. We do not go that far, but we do keep moments up through order ten. We find that including systematic co-moments 3 through 10 almost always causes SMB and HML to become insignificant, and always causes their *t*-statistics to drop dramatically. As a check, we re-run our tests using standard univariate moment estimates instead of co-moments. We find that SMB and HML usually remain significant even if standard moments 3 through 10 are included. Thus, it does not appear that our results are being driven by simply adding more explanatory variables.

In the next section we introduce the higher-order systematic co-moment equation. In Section III we present our data and perform the tests. Section IV summarizes and gives our conclusions.

II. Higher-Order Co-Moments

Assuming perfect markets, a risk-free asset, and homogeneous expectations, Rubinstein (1973) derives the following equation for $E(R_j)$, the expected return on security j:

$$E(R_j) = R_f + \sum_{n=2}^{\infty} \theta_{in} \cdot \sigma_{in}(R_j, \widetilde{W}_i), \qquad (1)$$

where σ_{in} is the n^{th} co-moment:

$$\sigma_{in}(R_j, \widetilde{W}_i) = E_i[R_j - E(R_j)] \cdot [\widetilde{W}_i - E_i(\widetilde{W}_i)]^{n-1},$$

$\widetilde{W}_i$ is individual i's future wealth, $\theta_{in} = \dfrac{-U_i^{(n)}}{(n-1)! \cdot E[U_i(\widetilde{W}_i)]}$, and $U_i(\widetilde{W}_i)$ is i's utility of wealth.

We can interpret σ_{in} and θ_{in} as individual i's measures of security risk and risk aversion, respectively. Rubinstein notes that an individual adjusts his portfolio until the expected return equals the risk-free rate plus a risk premium equal to a weighted sum of co-moments. Intuitively, investors are concerned about risk, and risk must be measured in terms of the entire probability distribution, which in turn can be measured with the moments of the distribution. Only in very special cases, such as quadratic utility or normality of returns, can we ignore the higher moments and focus on just mean and variance. For an individual security, the contribution to the risk of the portfolio is measured with co-moments.

For the case of separable cubic utility Rubinstein derives the market relation:

$$E(R_j) = R_f + \lambda_2 Cov(R_j, R_m) + \lambda_3 Cos(R_j, R_m, R_m) \tag{2}$$

where *Cos* is co-skewness, *m* denotes the market portfolio, and λ_2 and λ_3 are market measures of risk aversion. Kraus and Litzenberger (1976) derive an equation of the same form. They state that "it is trivial to extend the model to incorporate any number of higher moments" (p. 1087). We are willing to make whatever restrictive assumptions are needed to extend equation (2) to the case of *n* co-moments. For example, if we assume identical investors, then equation (1) implies the market relation:

$$E(R_j) = R_F + \sum_{i=2}^{N} \lambda_i b_{ij} \tag{3}$$

where b_{ij} is the i^{th} order systematic co-moment between R_j and R_m, and λ_i is the market measure of risk aversion for the i^{th} co-moment. This follows from Rubinstein's equation (5):

$$\frac{\sum_i E_i(R_j)}{I} = \frac{\sum_i E_i(R_k)}{I} + \sum_{n=2}^{\infty} \frac{\theta_{in} \cdot \sigma_{in}(R_j - R_k, \widetilde{W}_i)}{I}$$

where I is the number of individuals. Letting *k* be the risk-free asset, assuming homogeneous expectations, noting that each sum over *i* reduces to I times the summand, and equating $\widetilde{W}_i$ to a function of the future value of the market portfolio (since in equilibrium all assets must be held), we obtain equation (3). Equation (3) is the generalization of the security market line: Instead of one market co-moment, there are several.

These models suggest that only market-risk factors should matter to investors. Yet Fama and French find that their non-market SMB and HML factors are also important. In the next section we test whether the SMB and HML factors are independent risk factors in their own right or merely proxies for higher-order market factors.

III. Data and Empirical Tests

A. The Fama-French Model

We explore the cross-sectional return characteristics of portfolios based on size and also based on book-to-market value over the 1930 to 1998 sample period. Our empirical tests are in the spirit of Fama and MacBeth (1973). Fama and MacBeth test the CAPM with a two-pass procedure that first sorts stocks into portfolios based on historical beta estimates and then estimates the mean cross-sectional relationship between the portfolio returns and portfolio betas for each period. By sorting on beta they are able to maximize the cross-sectional variation in the variable of interest. We perform this same experiment for both Fama-French factors. At the end of each calendar year, we rank all ordinary common stocks included on the Center for Research in Security Prices (CRSP) file by market capitalization and divide the sample into 50 portfolios of equal size. The size portfolios increase from about 15 stocks per portfolio in the 1930s to about 140 stocks in the 1990s. Using this size-based portfolio definition, we compute a time series of equal-weighted and rebalanced portfolio returns. We subtract the 30-day Treasury bill yield to obtain the excess portfolio return, $r(j,t)$, where j denotes the portfolio and t denotes the time period. We repeat this procedure for the stratification using book-to-market value. For this sort, we use the universe of both CRSP and Compustat firms. The portfolios are sorted by the beginning-of-period book-to-market ratio and contain all CRSP and Compustat-listed ordinary common equities from 1970 to 1998 for all return intervals. To reiterate, by sorting independently by size and book-to-market value, we are able to maximize the variation in the risk factor at hand.

For each period, t, we estimate a cross-sectional regression of the period portfolio returns on the loadings on SMB and HML and systematic co-moment factor loadings as in the following equation:

$$r(j,t) = a_1 + a_{SMB}s(j,t) + a_{HML}h(j,t) + \sum_{i=2}^{n} a_i b(i,j,t) + e(j,t) \tag{4}$$

where $s(j,t)$ and $h(j,t)$ are the factor loadings for SMB and HML, respectively, and $b(i,j,t)$ is the ith systematic co-moment. For example, the 2nd systematic co-moment is the CAPM beta and the 3rd systematic co-moment is the Kraus and Litzenberger (1976) systematic co-skewness factor. Repeating this process for all periods in the sample period produces T sets of coefficient estimates. We then average the T estimates to produce a sample Fama-MacBeth coefficient estimate.

In Table 1 we report the results for the Fama-French three-factor (SMB, HML, and the 2nd systematic co-moment) model. Panel A presents results for size-sorted portfolios over the period 1930 to 1998 and Panel B has the results for book-to-market-ratio-sorted portfolios over the period 1970 to 1998. Since daily returns on the CRSP file do not begin until July 1962, the sample period in Panel A is shorter for the daily and weekly intervals (1965-1998). For each period, portfolio returns are regressed on three factor loadings: *b*, *s*, and *h*. These loadings are computed by regressing portfolio returns over the past five years on the SMB, HML, and market factors, respectively. We find in both panels that at least one of the Fama-French factor loadings, *s* and *h*, has significant explanatory power for portfolio returns for each of the return frequencies. Not surprisingly, *s* does better in Panel A and *h* does better in Panel B. Market beta has significant explanatory power every time, so all three factors appear to be important for explaining returns. We perform a Wald test of the joint significance of SMB and HML coefficients. The χ^2-statistic tests whether the coefficients on *s* and *h* are jointly different from zero. The Wald test is significant at the 1-percent level for all time intervals and sorting criteria.

The results in Keim (1983), Knez and Ready (1997), and Chan, Karceski, and Lakonishok (1998) emphasize the importance of January in understanding asset returns. We look (Panel C) at monthly returns of size-sorted portfolios by calendar month. The results are striking. For every month except January the coefficients for *b* and *h* have the normal sign and usually are significant, whereas the coefficient for *s* is insignificant. However, in January *b* and *h* are insignificant and have the opposite sign, whereas *s* is strongly significant.

B. The Distribution of Portfolio Returns

Table 2 provides summary statistics for the distributions of the portfolio returns used in Table 1. Panel A gives the results for size-sorted portfolios. The return-distribution skewness increases with the length of the return interval, increasing from -0.59 for daily intervals to 5.1 and 2.6 for quarterly and semi-annual intervals, respectively. The kurtosis statistic is consistently greater than the corrected standard normal distribution value of 0, varying from 4.5 for weekly returns to 58.9 for quarterly returns. Panel B gives the results for book-to-market-ratio-sorted portfolios. Again, skewness increases with the return interval, and kurtosis is always somewhat higher than that for the normal. To investigate the normality assumption, we provide test statistics for two standard normality tests. The Jarque-Bera statistic tests the third and fourth sample moments against those of a normal distribution. In Panel A, for all intervals except weekly, the Jarque-Bera statistic for the portfolio returns strongly rejects normality. In Panel B, this statistic does not reject normality except for daily returns. Portfolio return normality is, however, strongly rejected in every case by the Kolmogorov statistic. This statistic measures the maximum and minimum discrepancy between the normal cumulative density function and the empirical cumulative density function generated by the data. Because we have argued that skewness and kurtosis are not sufficient to describe the distribution, we consider the Kolmogorov statistic to be the appropriate one for our purposes.

C. Higher-Order Systematic Co-moments

Since returns are not normal, we expect higher-order co-moments to matter to risk-averse investors concerned about extreme outcomes. We compute systematic co-moment estimates of the 3rd to 10th order using the past five years of portfolio returns. Non-centered systematic co-moments *b(i,j,t)* are computed in the following manner:

$$b(i,j,t)=\frac{\sum_{\tau=1}^{T} r(j,t-\tau)\left(r(m,t-\tau)\right)^{i-1}}{\sum_{\tau=1}^{T}\left(r(m,t-\tau)\right)^{i}} \tag{5}$$

where *i* denotes the order of the co-moment and *r (m,t)* is the return for the CRSP value-weighted portfolio. (The CRSP portfolio covers NYSE and AMEX stocks until 1973 when NASDAQ stocks are also added.) We compute a series of systematic co-moment estimates from the 3rd to the 10th order for each of our five return intervals. To illustrate, for the monthly interval we compute monthly co-moment estimates using the past 60 months of return data. In this manner estimates are produced for each portfolio for each month from January 1930 to December 1998. The result is a monthly time series of co-moment estimates for each portfolio. The estimation procedure is similar for the other return intervals (e.g., the daily interval procedure produces a time series of daily co-moment estimates). Since the estimation interval is constant (five years), the daily interval co-moment estimates are computed from about 1,260 daily portfolio-return observations, whereas the semi-annual estimates are computed from only ten portfolio-return observations.

We also performed the experiments in this paper using centered co-moment estimates by demeaning each series by the sample mean. We report the results based on the non-centered series since centering causes the denominator of odd co-moment estimates to tend to zero so that the centered estimates can vary dramatically.

To get a sense of the correlation between the F-F loadings and the higher order co-moments, we regress the F-F loadings on the second- to tenth-order monthly systematic co-moment estimates. We find that the systematic co-moments can explain (loadings for) SMB and HML with fairly high R^2 (.86 to .92 for size-sorted portfolios and .81 to .93 for book-to-market-sorted portfolios). We can also explain SMB and HML with standard moments, but with less success: The R^2 are consistently lower. We find that R^2 increases as we add co-moments and also increases as we use co-moments instead of moments. This result motivates the following empirical question: Are SMB and HML true risk factors or do they merely proxy for higher-order systematic co-moments?

D. Empirical Tests

The mean coefficient estimates from equation (4) for the F-F factors and systematic co-moments are reported in Table 3 for each return frequency interval. Panel A reports the results using size sorting. For daily returns, the SMB loading remains significant, but its *t*-statistic drops significantly as co-moments are added, whereas the HML loading maintains sporadic weak significance. For weekly returns, SMB becomes insignificant when the tenth co-moment is added and HML becomes insignificant when the seventh is added. For monthly returns, both SMB and HML become insignificant when co-moments greater than the fourth order are included. For quarterly returns, the highly significant HML loading from Table 1 becomes insignificant or has the wrong sign when any higher-order co-moments are included. The same is usually true for semi-annual returns, although the HML loading returns to significance in some specifications. Except for the daily return interval, the X^2-statistic, which tests the joint significance of the SMB and HML coefficients, is significant only when few higher-order co-moments are included. Panel B presents the results for book-to-market sorting. The results are similar to those in Panel A.

Thus, in every case, the premia on the SMB and HML loadings become insignificant or much less significant as systematic co-moments are added.[1] The fact that the adjusted R^2 statistics generally rise as we add co-moments suggests that the additional regressors are adding explanatory power. In some cases co-skewness or co-kurtosis or both are sufficient to eliminate the significance of both SMB and HML. In other cases higher-order co-moments are required.

It might be argued that almost any set of variables would reduce the significance of the Fama-French factors if enough of them are included. To check this, we substitute standard univariate moments (skewness, kurtosis, etc.) for our systematic co-moments (co-skewness, co-kurtosis, etc.):

$$m(i,j,t) = \frac{1}{T}\sum_{\tau=1}^{T}\left(r(j,t-\tau)\right)^{i} \qquad (6)$$

[1] A lower significance level may be required as the rejection criterion for large samples, such as our daily (8,561) and weekly (1,774) observations. See Lindley (1957). Most of the F-F factor loadings that remain significant at the conventional levels of significance, such as 5% or 1%, become insignificant when we use the 0.1% level instead.

where *m(i,j,t)* is the *i*th order standard moment for portfolio *j* at time *t*. We re-run the regression replacing the co-moment estimates with the standard higher-order moment estimates over orders 3 through 10.

$$r(j,t) = a_1 + a_{SMB}s(j,t) + a_{HML}h(j,t) + a_2 b(2,j,t) + \sum_{i=3}^{10} a_i m(i,j,t) + e(j,t) \qquad (7)$$

The results, in Table 4, show that with size sorting the F-F factors maintain highly significant explanatory power for portfolio returns in every case (with the exception of the HML loading with the quarterly return specification) even when moments 3-10 are included. For book-to-market sorting, the SMB premium estimates are insignificant, but, with the exception of daily returns, the *s* estimates are also insignificant when no higher moments or co-moments are included (see Table 1, panel B). The HML estimates remain consistently significant when the higher moments are included. Thus, systematic co-moments reduce the significance of the F-F factors, but the standard moments do not.

E. Errors-in-Variables Concerns

One important concern with our empirical approach is that the two-pass Fama-MacBeth method may be biased since the right-hand side variables in the second-pass cross-sectional regression are the estimates from the first-pass time-series regression (Shanken (1992), Kim (1995, 1997), and Jagannathan and Wang (1998)). To test for the errors-in-variables (EIV) bias we recalculate all standard errors using the Shanken adjustment.[2] The adjusted t-statistics

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[2] The adjusted covariance matrix is defined as: $adj.\text{var}(a) = V\left[1 + \bar{a}'(Z)^{-1}\bar{a}\right] + Z$ where V is the k-factor-by-k-factor covariance matrix of the monthly demeaned Fama-MacBeth coefficient estimates, $\bar{a}$ is the T-periods-by-k matrix of mean coefficient estimates, and Z is the k-by-k covariance matrix of monthly risk factors. The respective risk factors are the standard Fama-French SMB and HML factors, the CRSP value-weighted market-portfolio

are slightly lower for specifications with just a few higher-order co-moments. However, when additional higher-order co-moments are added, the standard errors increase substantially, causing the t-statistics to approach zero.[3] We are concerned that the Shanken adjustment biases the test results too much against the Fama-French factors. Because of the way our higher-order right-hand side variables are created, the adjustment appears to be inappropriate for specifications that include such variables. Omitting the Shanken adjustment appears to inflate the explanatory power of the Fama-French factors and bias our tests against the Rubinstein hypothesis.[4]

We perform an additional robustness test to investigate further EIV concerns in our test specifications. To test for sensitivity to the estimation window, we use the original weekly and monthly return data to generate two different sets of loading estimates with the return data—one from the early part of the rolling estimation window (years t-5 to t-3) and the other from

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return less the 30-day Treasury Bill yield (RMRF) for the covariance factor, and RMRF raised to the i-1 power for the higher-order co-factors.

[3] The increase in standard errors is due to the mechanics of the adjustment. The covariance matrix, Z, is estimated as F'*F, where F is the matrix of demeaned factors. Since the higher-order factors in F are estimates as excess portfolio returns raised to higher powers, they have very small values. When they are multiplied against each other, the elements of the Z matrix become even smaller as higher-order factors are added. Since the multiplicative factor in the adjustment includes the inverse of Z, the adjusted covariance matrix becomes extremely large as higher-order factors are added. For the size-sorted weekly return interval, for example, the unadjusted t-statistic on the *s* coefficient with the Fama-French specification is 6.28 while the Shanken-adjusted t-statistic is 3.07. With co-moments 3 through 10 included, the unadjusted t-statistic is 1.48 while the respective adjusted t-statistic drops to 0.0000000000031. The other test specifications experience similar large increases in the Shanken-adjusted standard errors.

[4] We also considered the Kim (1995, 1997) EIV correction method, which is defined for a setup where one of the regressors is measured with error (e.g., beta) and the other right-hand side variables are error-free. Our experiment setup, however, involves multiple right-hand side variables measured with error (the SMB and HML loadings, as well as beta and the higher-order co-moments, are all estimates). Extending the EIV correction procedure to a multivariate framework appeared to be sufficiently problematic that we were unable to account explicitly for the correction.

the late part of the rolling estimation window (years t-2 to t-1). We then use the two additional sets of right-hand side variables to re-estimate the regressions in the paper: once with the loadings based on the early-window returns and once with the loadings based on the late-window returns. If our results are based on errors in the estimates, we would expect that using two unique sets of right-hand side variables based on the different estimation windows should provide a basis to test for the importance of such bias. We find that the coefficient estimates and pattern in test statistics are roughly consistent across different estimation periods. In no case are the results substantially different from what is reported in the paper. The ability of the F-F factor loadings to maintain explanatory power does not appear to be sensitive to the right-hand side variable estimation window. We therefore conclude that although our estimates are likely to include some EIV bias, we do not believe that the bias is large enough to negate our overall conclusions.

F. Robustness Checks

So far, all the returns have been computed discretely. We tried continuous returns. This change caused the skewness to change signs, but otherwise the results were similar: Normality can be rejected, and F-F factors start out significant but become insignificant when co-moments (but not moments) are included. We also tried the CRSP equal-weighted portfolio as the market proxy and obtained similar results.

As another robustness check, we sorted stocks based on momentum (performance over the previous six months). The results were very similar to those for book-to-market sorting.

We also looked at January returns, and got similar results. The Kolmogorov statistic strongly rejects normality for January returns. Introducing co-moments gradually reduces the *t*-statistic for *s* from 7 (Table 1, Panel C) to 2.077 with co-moments 3 through 10 included.

Introducing moments instead of co-moments is a bit less successful at reducing the *t*-statistic (t=2.675 with moments 3 through 10 included).

If we had found a case where returns were normal, we would have expected the F-F factors to be insignificant. Note that the converse need not hold: Insignificance of the F-F factors does not imply returns are normal. We also tested if returns are lognormal. Because continuous-compounding factors are exponentials, continuous returns are additive, so we might expect the Central Limit Theorem, under certain assumptions, to hold. Then continuous returns would be asymptotically normal, and discrete returns would be asymptotically lognormal. However, we found we could reject lognormality as well as normality.

We also tried sorting on the Kolmogorov statistic for normality. One might suppose that the sub-sample for which normality cannot be rejected would obey the CAPM, but this is not what we found. Perhaps investors did not expect this sub-sample to exhibit normality ex ante and so the F-F factors were priced anyway.

We note that, whereas our results seem to be robust, the F-F factors seem to lack robustness: one or the other is insignificant in almost every case (Table 1) and adding co-moments causes the F-F factors in most cases to become insignificant (Table 3).

Finally, note that, for some cases, the coefficients for SMB and HML do not change much as co-moments are added: The *t*-statistics decline because the estimated standard deviations increase. Thus, one may be tempted to argue that our results are driven by imprecision in estimation caused by adding co-moments. However, two facts strongly suggest this is not the case. First, for the longer intervals, many of the coefficients (SMB and HML for quarterly and semi-annual) change signs. Second, using moments instead of co-moments does not lead to a decline in significance levels.

IV. Summary and Conclusions

We can reject normality of returns for daily, weekly, monthly, quarterly and semi-annual intervals. In the absence of normality, investors should be very concerned with the shape of the tails of the distribution of portfolio returns, which can be measured with a set of higher-order co-moments. Our results suggest that the Fama-French factors proxy for higher-order co-moments, as the F-F loadings generally become insignificant when higher-order systematic co-moments are included in cross-sectional regressions for portfolio returns. Thus, we find evidence for a model of the sort given in Rubinstein (1973), i.e., measuring risk requires more than just measuring covariance. Higher-order co-moments matter to risk-averse investors concerned about extreme outcomes.

It is theoretically satisfying to have a logical explanation for the success of SMB and HML at explaining security returns. In principle, we would expect higher-order co-moments to be better for such practical matters as measuring portfolio performance. However, it is conceivable that the SMB and HML loadings are such good proxies for the higher-order co-moments that, given problems of estimating higher-order co-moments, the Fama-French factors could be superior in actual use.

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TABLE 1 Fama-MacBeth regression results for the Fama-French Model

Panel A. Size-sorted portfolios

Return frequency	Periods	*b*	*s*	*h*	Mean adj r^2 [Joint test]
Daily	8561	0.0009**	0.0003**	-0.0001	0.189
		(6.66)	(4.37)	(-0.74)	[20.0**]
Weekly	1774	0.0104**	0.0021**	-0.0025**	0.288
		(13.57)	(6.28)	(-5.20)	[59.5**]
Monthly	828	0.0229**	0.0020*	-0.0089**	0.337
		(8.35)	(1.68)	(-6.82)	[57.9**]
Quarterly	276	0.0492**	0.0038	-0.0334 **	0.417
		(6.11)	(0.78)	(-7.29)	[57.1**]
Semi-annual	138	0.0505**	-0.0040	-0.0390**	0.369
		(3.56)	(-0.50)	(-5.01)	[25.3**]

Panel B. Book-to-market-ratio-sorted portfolios

Return frequency	Periods	*b*	*s*	*h*	Mean adj r^2 [Joint test]
Daily	7075	0.0020**	0.0004**	-0.0011**	0.123
		(10.90)	(2.81)	(-7.68)	[77.0**]
Weekly	1514	0.0132**	-0.0011	-0.0064**	0.219
		(15.70)	(-1.91)	(-13.96)	[201.0**]
Monthly	348	0.0343**	-0.0008	-0.0303**	0.391
		(8.93)	(-0.36)	(-16.07)	[283.6**]
Quarterly	116	0.0626**	-0.0028	-0.0721 **	0.528
		(5.47)	(0.47)	(-11.72)	[137.5**]
Semi-annual	58	0.1160**	0.0102	-0.1260**	0.540
		(6.37)	(1.08)	(-8.77)	[71.2**]

TABLE 1 (Continued) Fama-MacBeth regression results for the Fama-French Model

Panel C. Monthly returns of size-sorted portfolios by calendar month

Month	Periods	*b*	*s*	*h*
January	69	-0.0154	0.0365**	0.0170
		(-1.25)	(7.00)	(3.20)
February	69	0.0058	0.0041	-0.0067*
		(0.90)	(1.38)	(-1.91)
March	69	0.0220**	-0.0003	-0.0076**
		(3.01)	(-0.09)	(-2.70)
April	69	0.0092	-0.0025	-0.0077**
		(1.17)	(-0.79)	(-2.50)
May	69	0.0228**	0.0003	-0.0044
		(2.52)	(0.06)	(-0.72)
June	69	0.0220**	-0.0074	-0.0035
		(2.61)	(-2.26)	(-0.99)
July	69	0.0258**	0.0011	-0.0089 **
		(2.39)	(0.38)	(-2.72)
August	69	0.0214**	0.0016	-0.0077*
		(3.33)	(0.30)	(-2.08)
September	69	0.0202**	0.0048	-0.0093*
		(2.38)	(1.18)	(-2.05)
October	69	0.0276**	-0.0073	-0.0147 **
		(2.91)	(-2.23)	(-3.78)
November	69	0.0502**	-0.0016	-0.0231 **
		(5.25)	(-0.46)	(-5.05)
December	69	0.0625**	-0.0055	-0.0295**
		(4.98)	(-1.48)	(-4.94)

The sample consists of daily, weekly, monthly, quarterly, and semi-annual returns of 50 equal-sized portfolios. In Panel A and C, the portfolios are sorted by beginning-of-period size and contain all CRSP listed ordinary common equities from 1930 to 1998 for the monthly, quarterly, and semi-annual return intervals and from 1965 to 1998 for the daily and weekly return intervals. In Panel B, the portfolios are sorted by beginning-of-period book-to-market ratio and contain all CRSP and Compustat-listed ordinary common equities from 1970 to 1998 for all return intervals. For each calendar period, portfolio returns are regressed on three factor loadings: *b*, *s*, and *h*. These loadings are computed by regressing portfolio returns over the past five years on the market, SMB, and HML factors, respectively. SMB represents the return on a portfolio of small stocks less the return on a portfolio of large stocks while HML represents the return on a portfolio of high book-to-market-value stocks less the return on a portfolio of low book-to-market-value stocks. The mean coefficient estimates across the sample period are reported with their *t*-statistics. The joint test is a Wald test of the joint significance of the *s* and *h* estimates.
* and ** denote one-tail significance at the 5-percent and 1-percent level, respectively.

TABLE 2 Summary statistics of portfolio returns

Panel A. Size-sorted portfolios

	Daily	Weekly	Monthly	Quarterly	Semi-annual
Number of portfolio-period observations	428,050	88, 700	41,393	13,798	6,899
Mean	0.0006	0.0022	0.0103	0.0335	0.0641
Variance	0.0001	0.0005	0.0072	0.0412	0.0706
Skewness	-0.5905	-0.3963	2.318	5.059	2.5872
Kurtosis	11.970	4.541	25.66	58.86	20.25
Jarque-Bera statistic	1457597.5**	11098.2	922517**	1852337**	93228.4**
Kolmogorov statistic	0.0591**	0.0459**	0.0980**	0.1347**	0.0914**

Panel B. Book-to-market-ratio-sorted portfolios

	Daily	Weekly	Monthly	Quarterly	Semi-annual
Number of portfolio-period observations	353,750	75,700	17,400	5,800	2,900
Mean	0.00070	0.0022	0.0074	0.0228	0.0466
Variance	0.00007	0.0006	0.0044	0.0214	0.0529
Skewness	-0.7913	-0.3951	-0.0124	0.3708	0.7880
Kurtosis	13.388	5.120	3.218	1.1643	1.7794
Jarque-Bera statistic	1627167.3**	16142.5	35.06	947.3	480.1
Kolmogorov statistic	0.0604**	0.0499**	0.0443**	0.0471**	0.0624**

The sample consists of daily, weekly, monthly, quarterly, and semi-annual returns of 50 equal-sized portfolios. In Panel A, the portfolios are sorted by beginning-of-period size and contain all CRSP listed ordinary common equities from 1930 to 1998 for the monthly, quarterly, and semi-annual return intervals and from 1965 to 1998 for the daily and weekly return intervals. In Panel B, the portfolios are sorted by beginning-of-period book-to-market ratio and contain all CRSP and Compustat-listed ordinary common equities from 1970 to 1998 for all return intervals.

* and ** denote significance at the 5-percent and 1-percent level, respectively.

TABLE 3 Fama-MacBeth regression results for Fama-French (F-F) factors and systematic co-moments

Panel A. Size-sorted portfolios

System. co-moments	Daily (periods=8561)			Weekly (periods=1774)			Monthly (periods=828)			Quarterly (periods =276)			Semi-annual (periods=138)		
	s	*h*	adj r^2 [Joint]	*s*	*h*	adj r^2 [Joint]	*s*	*h*	adj r^2 [Joint]	*s*	*h*	adj r^2 [Joint]	*s*	*h*	adj r^2 [Joint]
2^{nd} to 3^{rd}	.0006**	-.0001	0.200	0.003**	-.001**	0.298	0.005**	-0.002*	0.361	0.002	-0.005	0.463	-0.011	-0.012	0.481
	(7.14)	(-1.00)	[3.77*]	(8.06)	(-2.67)	[69.0**]	(3.83)	(-1.70)	[19.0**]	(0.24)	(-0.83)	[2.06]	(-1.30)	(-1.48)	[3.45]
2^{nd} to 4^{th}	.0007**	-.0003*	0.206	0.002**	-.002**	0.304	0.003*	-0.003*	0.367	0.005	-0.004	0.471	0.003	-0.021*	0.488
	(6.96)	(-1.98)	[1.96]	(4.75)	(-3.63)	[30.8**]	(1.89)	(-1.89)	[3.23]	(0.69)	(-0.51)	[4.04]	(0.31)	(-2.28)	[3.64]
2^{nd} to 5^{th}	.0007**	-.0003*	0.215	0.002**	-.002**	0.312	-0.001	-0.002	0.376	-0.000	0.003	0.482	-0.030	-0.026*	0.503
	(6.79)	(-1.86)	[1.79]	(5.07)	(-3.63)	[33.0**]	(-0.38)	(-0.86)	[0.78]	(-0.05)	(0.39)	[2.46]	(-2.19)	(-1.92)	[8.05*]
2^{nd} to 6^{th}	.0008**	-.0004*	0.220	0.002**	-0.002*	0.317	0.001	-0.002	0.379	-0.003	0.017	0.488	-.0003	0.007	0.508
	(6.23)	(-1.96)	[1.64]	(4.78)	(-1.99)	[22.7**]	(0.26)	(-0.53)	[0.44]	(-0.30)	(2.19)	[1.50]	(-0.01)	(0.28)	[1.97]
2^{nd} to 7^{th}	.0008**	-.0004	0.226	0.002**	-0.001	0.321	-0.001	-0.001	0.385	-0.011	0.007	0.494	-0.058	-0.046	0.510
	(5.27)	(-1.49)	[1.54]	(4.59)	(-1.49)	[19.3**]	(-0.25)	(-0.39)	[1.19]	(-0.90)	(0.65	[0.51]	(-1.23)	(-0.87)	[5.88]
2^{nd} to 8^{th}	.0007**	-.0004	0.233	0.002**	-0.001	0.327	-0.001	-0.001	0.390	-0.008	0.007	0.496	-0.033	-0.037	0.507
	(3.61)	(-1.56)	[1.53]	(3.03)	(-0.78)	[8.92*]	(-0.43)	(-0.33)	[3.27]	(-0.58)	(0.57)	[0.11]	(-0.62)	(-0.61)	[2.26]
2^{nd} to 9^{th}	.0005*	-.0005*	0.236	0.002**	-0.001	0.331	0.001	-0.001	0.391	-0.016	0.020	0.496	0.013	-0.071	0.499
	(2.33)	(-1.98)	[1.50]	(2.50)	(-0.93)	[6.59*]	(0.39)	(-0.18)	[0.10]	(-1.15)	(1.53)	[1.03]	(0.22)	(-1.12)	[2.63]
2^{nd} to 10^{th}	.0007**	-.0003	0.237	0.001	-0.000	0.335	0.003	-0.002	0.394	-0.029	0.029	0.488	0.049	-0.204*	0.489
	(3.13)	(-1.07)	[1.47]	(1.48)	(-0.26)	[2.85]	(0.69)	(-0.30)	[0.37]	(-1.81)	(2.09)	[3.13]	(0.65)	(-1.68)	[2.36]

TABLE 3 (Continued) Fama-MacBeth regression results for Fama-French (F-F) factors and systematic co-moments

Panel B. Book-to-market-ratio-sorted portfolios

	Daily (periods=7075)			Weekly (periods=1514)			Monthly (periods=348)			Quarterly (periods =116)			Semi-annual (periods=58)		
System. co-moments	*s*	*h*	adj r^2 [Joint]	*s*	*h*	adj r^2 [Joint]	*s*	*h*	adj r^2 [Joint]	*s*	*h*	adj r^2 [Joint]	*s*	*h*	adj r^2 [Joint]
2nd to 3rd	.0009**	-.001**	0.133	0.002**	-.003**	0.239	0.006**	-.011**	0.440	-0.013	-0.024**	0.594	-0.018	-0.013	0.666
	(5.27)	(-7.26)	[1.93]	(4.09)	(-7.57)	[72.7**]	(2.83)	(-6.15)	[50.0**]	(-2.56)	(-4.38)	[31.7**]	(-2.07)	(-1.01)	[6.44*]
2nd to 4th	.0010**	-.001**	0.136	0.002**	-.003**	0.247	0.004	-.008**	0.447	-0.008	-0.024**	0.601	-0.017	-0.014	0.676
	(5.65)	(-6.68)	[1.73]	(2.78)	(-5.91)	[33.4**]	(1.53)	(-3.97)	[16.3**]	(-1.09)	(-3.76)	[23.4**]	(-0.81)	(-0.92)	[2.62]
2nd to 5th	.0009**	-.001**	0.140	0.001*	-.002**	0.252	-0.002	-.007**	0.453	-0.020	-0.009	0.617	-0.055	-0.017	0.687
	(4.66)	(-5.27)	[1.64]	(1.71)	(-3.43)	[13.7**]	(-0.65)	(-3.13)	[17.2**]	(-2.68)	(-1.29)	[19.0**]	(-1.92)	(-0.95)	[5.15]
2nd to 6th	.0010**	-.001**	0.142	0.001	-0.002*	0.256	-0.003	-0.001	0.460	-0.024	0.0004	0.625	-0.121	0.015	0.701
	(4.84)	(-2.88)	[1.58]	(1.62)	(-1.99)	[5.05]	(-0.94)	(-0.44)	[5.54]	(-1.66)	(0.32)	[9.40**]	(-1.70)	(0.20)	[4.18]
2nd to 7th	.0004*	-.001**	0.146	0.001	.0002	0.261	-0.004	-0.001	0.464	-0.038	0.008	0.631	-0.136	0.017	0.707
	(1.94)	(-3.33)	[1.45]	(1.51)	(0.20)	[1.04]	(-1.33)	(-0.21)	[7.32*]	(-2.46)	(0.63)	[13.9**]	(-2.16)	(0.34)	[9.29**]
2nd to 8th	.0006*	-.001**	0.149	0.0005	-.0001	0.268	-0.004	-0.001	0.469	0.0008	-0.017	0.635	-0.203	0.053	0.711
	(2.28)	(-3.47)	[1.36]	(0.56)	(-0.08)	[0.17]	(-0.84)	(-0.16)	[6.04*]	(0.04)	(-0.98)	[2.55]	(-2.26)	(-0.66)	[4.07]
2nd to 9th	.0006*	-.001**	0.148	0.0002	0.002	0.270	-0.004	0.005	0.468	-0.037	0.022	0.642	-0.325	0.111	0.706
	(2.30)	(-2.38)	[1.35]	(0.23)	(1.37)	[1.54]	(-0.83)	(0.94)	[0.69]	(-1.47)	(1.21)	[2.49]	(-2.17)	(0.96)	[5.43]
2nd to 10th	.0008**	-.0005	0.143	0.0004	0.0019	0.270	-0.001	0.001	0.467	-0.040	0.025	0.637	-0.312	0.078	0.704
	(2.72)	(-1.57)	[1.29]	(0.41)	(1.41)	[0.03]	(-0.23)	(0.20)	[1.11]	(-1.52)	(1.17)	[1.18]	(-2.03)	(0.65)	[10.2**]

The sample consists of daily, weekly, monthly, quarterly, and semi-annual returns of 50 equal-sized portfolios. In Panel A, the portfolios are sorted by beginning-of-period size and contain all CRSP listed ordinary common equities from 1930 to 1998 for the monthly, quarterly, and semi-annual return intervals and from 1965 to 1998 for the daily and weekly return intervals. In Panel B, the portfolios are sorted by beginning-of-period book-to-market ratio and contain all CRSP and Compustat-listed ordinary common equities from 1970 to 1998 for all return intervals. The F-F factor loadings *s* and *h* are computed from the three-factor model estimates using portfolio returns over the past five years. The systematic co-moments are estimates using the same rolling five-year portfolio return data with non-centered return data. For each period the portfolio return is regressed on the F-F loadings and the respective number of systematic co-moments. The mean coefficient estimates across the sample period are reported. The joint test is a Wald test of the joint significance of the *s* and *h* estimates.
* and ** denote one-tail significance at the 5-percent and 1-percent level, respectively, with respect to the empirical sign on SMB and HML generated in the tests in Table 1.

TABLE 4 Fama-MacBeth regression results for Fama-French (F-F) factors and standard moments of order 3 through 10

	Size-sorted portfolios			Book-to-market-ratio-sorted portfolios		
Return frequency	*s*	*h*	Mean adj r^2 [Joint test]	*s*	*h*	Mean adj r^2 [Joint test]
Daily	0.0005** (4.23)	-0.0005** (-2.74)	0.254 [1.69]	-0.0001 (-0.57)	-0.0009** (-5.33)	0.191 [31.5**]
Weekly	0.0020** (4.18)	-0.0026** (-4.70)	0.351 [31.5**]	-0.0009 (-1.01)	-0.0046** (-8.39)	0.314 [83.5**]
Monthly	0.0075** (4.56)	-0.0034* (-1.97)	0.436 [18.5**]	0.0004 (0.13)	-0.0145** (-7.98)	0.508 [45.8**]
Quarterly	0.0219** (2.48)	0.0044 (0.43)	0.534 [7.59*]	0.0030 (0.38)	-0.0317** (-6.25)	0.676 [18.4**]
Semi-annual	0.0245** (2.50)	-0.0197** (-2.68)	0.544 [14.4**]	-0.0224 (-1.23)	-0.0407** (-4.01)	0.755 [16.6**]

The sample consists of daily, weekly, monthly, quarterly, and semi-annual returns of 50 equal-sized portfolios. As denoted, the portfolios are sorted in two ways. The size-sorted portfolios are sorted by beginning-of-period size and contain all CRSP-listed ordinary common equities from 1930 to 1998 for the monthly, quarterly, and semi-annual return intervals and from 1965 to 1998 for the daily and weekly return intervals. The book-to-market-ratio-sorted portfolios are sorted by beginning of period book-to-market ratio and contain all CRSP and Compustat listed ordinary common equities from 1970 to 1998 for all return intervals. The F-F factor loadings *s* and *h* are computed from the three-factor model estimates using portfolio returns over the past five years. The standard moments are estimates using the same rolling five-year portfolio return data. For each period the portfolio return is regressed on the F-F factor loadings, beta, and moments three through ten. The mean coefficient estimates across the sample period are reported. The joint test is a Wald test of the joint significance of the *s* and *h* estimates.

* and ** denote one-tail significance at the 5-percent and 1-percent level, respectively.