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Duration and Convexity

The price of a bond is a function of the promised payments and the market required rate of return. Since the promised payments are fixed, bond prices change in response to changes in the market determined required rate of return. For investor's who hold bonds, the issue of how sensitive a bond's price is to changes in the required rate of return is important. There are four measures of bond price sensitivity that are commonly used. They are Simple Maturity, Macaulay Duration (effective maturity), Modified Duration, and Convexity. Each of these provides a more exact description of how a bond price changes relative to changes in the required rate of return.

Maturity

Simple maturity is just the time left to maturity on a bond. We generally think of 5-year bonds or 10-year bonds. It is straightforward and requires no calculation. The longer the time to maturity the more sensitive a particular bond is to changes in the required rate of return. Consider two zero coupon bonds, each with a face value of $1,000. Bond A matures in 10 years and has a required rate of return of 10%. The price[1] of Bond A is $376.89, where

$$P_A = \frac{\$1{,}000}{(1+.10/2)^{20}} = \$376.89$$

Bond B has a maturity of 5 years and also has a required rate of return of 10%. Its price is $613.91 or

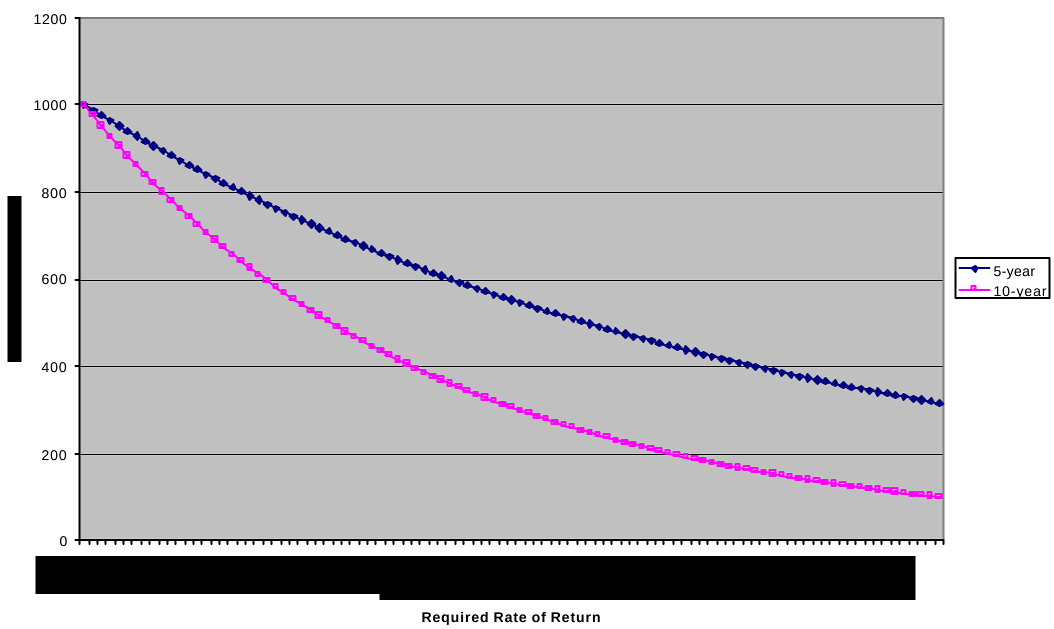
$$P_B = \frac{\$1{,}000}{(1+.10/2)^{10}} = \$613.91$$

[1] By convention, zero coupon bonds are compounded on a semi-annual basis. Since almost all US bonds have semi-annual coupon payments, this note will always assume semi-annual compounding unless otherwise noted.

If the required rate of return for each bond was to increase by 100 basis points to 11%, the prices would then be $342.73 for Bond A and $585.43 for Bond B. This translates into a -9.1% change in price for Bond A and -4.6% for Bond B.

Just from the pricing formulation, it is clear that any change in interest rates will have a much

Figure 1.
Comparison of 5-year and 10-year Bonds



greater impact on Bond A than Bond B. This is reinforced in Figure 1, where the price curve for the 10-year bond (Bond A) is much steaper than that for the 5-year bond (Bond B). Thus, for zero coupon bonds simple maturity can be used to compare price sensitivity.

Macaulay Duration (Effective Maturity)

The relationship between price and maturity is not as clear when you consider non-zero coupon bonds. For a coupon-paying bond, many of the cash flows occur before the actual maturity of the bond and the relative timing of these cash flows will affect the pricing of the bond. In order to deal with this, Frederick Macaulay[2] in 1938 suggested that investors use the effective maturity of a bond as a measure of interest rate sensitivity. He called this duration and defined it as a value-weighted average of the timing of the cash flows. The easiest way to see this is to use an example. Consider a six-year bond with face value of $1,000, and a 6.1% coupon rate (semi-annual payments). If the current yield to maturity is 10%, the value of the bond is found by discounting each of the semi-annual payments. This is shown in Exhibit 1.

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[2] Fredick Macaulay, *Some Theoretical Problems Suggested by the Movement of Interest Rates, Bond Yields, and Stock Prices in the United States since 1856* (New York: National Bureau of Economic Research, 1938).

Exhibit 1

t	Cash Flow	Pres. Value Factor	Pres. Value of Cash Flow
1	$ 30.50	0.9524	$ 29.05
2	30.50	0.9070	27.66
3	30.50	0.8638	26.35
4	30.50	0.8227	25.09
5	30.50	0.7835	23.90
6	30.50	0.7462	22.76
7	30.50	0.7107	21.68
8	30.50	0.6768	20.64
9	30.50	0.6446	19.66
10	30.50	0.6139	18.72
11	30.50	0.5847	17.83
12	1,030.50	0.5568	573.82

Macaulay Duration takes the present value of each payment and divides it by the total bond price, P. By doing this, one has a percentage, w_t, of the total bond value that is received in each period, t.

$$w_t = \frac{C_t / (1+y)^t}{P}$$

The duration or effective maturity for the bond could then be estimated by multiplying the weight, w_t, times the time, t and then summing all of the weighted values, or

$$Duration = \sum_{t=1}^{T} \frac{C_t / (1+y)^t}{P} \cdot t = \sum_{t=1}^{T} w_t \cdot t \, .$$

This measure takes into account the relative timing of the cash flows. Calculation of the Macaulay Duration measure is fairly straightforward but can be somewhat tedious[3]. Exhibit 2 shows how a semi-annual duration for the example shown above would be calculated.

[3] Excel offers a worksheet function DURATION(.), which calculates the Macaulay Duration.

Exhibit 2

t	Cash Flow	Pres. Value Factor	Pres. Value of Cash Flow	Weight	t x weight
1	$ 30.50	0.9524	$ 29.05	3.51%	0.0351
2	30.50	0.9070	27.66	3.34%	0.0669
3	30.50	0.8638	26.35	3.19%	0.0956
4	30.50	0.8227	25.09	3.03%	0.1213
5	30.50	0.7835	23.90	2.89%	0.1445
6	30.50	0.7462	22.76	2.75%	0.1651
7	30.50	0.7107	21.68	2.62%	0.1834
8	30.50	0.6768	20.64	2.50%	0.1997
9	30.50	0.6446	19.66	2.38%	0.2139
10	30.50	0.6139	18.72	2.26%	0.2264
11	30.50	0.5847	17.83	2.16%	0.2371
12	1,030.50	0.5568	573.82	69.37%	8.3246
	Bond Value		$ 827.17	Semi-Annual Duration	10.014

The semi-annual duration for this bond is 10.014 six-month periods. We usually use annual duration and we annualize the semi-annual duration simply by dividing by 2 (the number of six month periods in a year). In this case, the annualized duration would be 5.007 years. Note that the Macaulay Duration for a 5-year zero coupon bond is the same as the simple maturity, 5.0 years. Hence, we can expect that the original 6-year, 6.1% coupon bond when interest rates change to behave in a manner similar to a 5-year zero coupon bond, since their effective maturity (Macaulay Duration) is essentially the same.

Modified Duration

If we want a more direct measure of the relationship between changes in interest rates and changes in bond prices, we can use Modified Duration. Modified Duration, D, is defined as the following

$$D = -\frac{1}{P} \cdot \frac{\Delta P}{\Delta y}$$

where P is the bond price, ΔP is the change in bond price and Δy is the change in the required rate of return (yield to maturity). For those with a math background, $\Delta P / \Delta y$ is the first derivative of the bond price with respect to yield to maturity. The basic pricing formulation for bonds is

$$P = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t}$$

where C_t is the cash payment received in time period t and y is the semi-annual yield to maturity. Taking the derivative of P with respect toy,

$$\frac{\Delta P}{\Delta y} = -\frac{1}{(1+y)} \cdot \sum_{t=1}^{T} t \cdot \frac{C_t}{(1+y)^t}.$$

Inserting this into the formula for Modified Duration yields,

$$D = -\frac{1}{P} \cdot \left[-\frac{1}{(1+y)} \cdot \sum_{t=1}^{T} t \cdot \frac{C_t}{(1+y)^t} \right]$$

Rearranging the above slightly,

$$D = \frac{1}{(1+y)} \cdot \sum_{t=1}^{T} t \cdot \frac{C_t / (1+y)^t}{P}.$$

Comparing this to the definition of Macaulay Duration and using that definition we can write Modified Duration as

$$\text{Modified Duration} = D = \frac{1}{(1+y)} \text{ Macaulay Duration}$$

While it is easy calculate Modified Duration once you have Macaulay Duration the interpretations of the two are quite different. Macaulay Duration is an average or effective maturity. Modified Duration really measures how small changes in the yield to maturity affect the price of the bond. In fact, from the definition of Modified Duration we can write the following relationship:

$$\frac{\Delta P}{P} = -D \cdot \Delta y$$

or % change in bond price = - Modified Duration times the change in yield to maturity.

For example, the six-year 6.1% coupon bond above had a yield to maturity of 10% and a semi-annual Macaulay Duration of 10.014 (5.007 annual Macaulay Duration). The Modified Duration of this bond is $\frac{10.014}{(1+.05)}$ or 9.537 on a semi-annual basis or $\frac{9.537}{2}$ = 4.77 years on an annual basis[4]. Assuming that the yield to maturity of 10% increases by 25 basis points to 10.25%, based on the Modified Duration of 4.77 years the price of the bond should change by $\frac{\Delta P}{P} = -D \cdot \Delta y = -4.77 \cdot (.25\%) = -1.19\%$. The bond price should drop by 1.19% from $827.17 to $817.31 ($827.17*(1-.0119) = $817.31). The actual calculated price at a yield to maturity of 10.25% is $817.38.

Exhibit 3 shows the Modified Duration price change and the actual calculated price change for different changes in yield to maturity.

Exhibit 3

Bond Data

Coupon =	6.1%
Maturity=	6 years
Face Value =	$1,000
Yield to Maturity=	10%
Price =	$827.17
Modifed Duration=	4.770

New Yield to Maturity	Change in Yield	-D*Change in Yield	Predicted Price	Actual % change*	Actual Price**	Difference***
12.00%	2.00%	-9.54%	748.26	-9.01%	752.68	4.42
11.75%	1.75%	-8.35%	758.12	-7.94%	761.52	3.40
11.50%	1.50%	-7.16%	767.99	-6.85%	770.50	2.52
11.25%	1.25%	-5.96%	777.85	-5.75%	779.61	1.76
11.00%	1.00%	-4.77%	787.71	-4.63%	788.85	1.13
10.75%	0.75%	-3.58%	797.58	-3.50%	798.22	0.64
10.50%	0.50%	-2.39%	807.44	-2.35%	807.73	0.29
10.25%	0.25%	-1.19%	817.31	-1.18%	817.38	0.07
10.00%	0.00%	0.00%	827.17	-0.00%	827.17	-
9.75%	-0.25%	1.19%	837.03	1.20%	837.10	0.07
9.50%	-0.50%	2.39%	846.90	2.42%	847.18	0.28
9.25%	-0.75%	3.58%	856.76	3.66%	857.40	0.64
9.00%	-1.00%	4.77%	866.63	4.91%	867.78	1.15
8.75%	-1.25%	5.96%	876.49	6.18%	878.31	1.82
8.50%	-1.50%	7.16%	886.35	7.47%	889.00	2.64
8.25%	-1.75%	8.35%	896.22	8.79%	899.84	3.62
8.00%	-2.00%	9.54%	906.08	10.12%	910.84	4.76

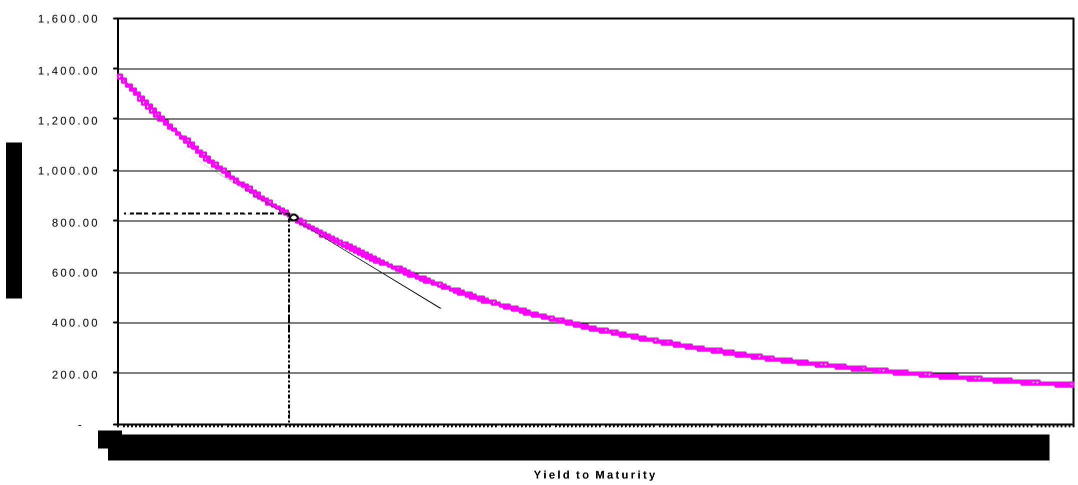
* Actual % change is based on the calculated price relative to the price of $827.17.
** Actual price is the calculated price based on the yield to maturity.
*** Difference is Actual Price - Predicted Price.

Modified Duration assumes that the price changes are linear with respect to changes in the yield to maturity. From Exhibit 3, the true relationship between the bond's price and the yield to maturity is not linear. The Column with the differences is always positive and increases

[4] If the original compounding basis on the bond was semi-annual, the Modified Duration must first be calculated on a semi-annual basis and then annualized. You can not use the annual Macaulay Duration to calculate the Modified Duration.

as we move away from a yield to maturity of 10%. The actual relationship between the bond price and the yield to maturity is shown in Figure 2.

Figure 2.
Bond Prices
6-year, 6.1% Coupon (paid semi-annually) Rate, Face value = $1,000



The curved line is the actual price curve. The straight line is the price relationship using Modified Duration. Everywhere the actual price curve is above the Modified Duration relationship. This is exactly what we saw in Exhibit 3. The difference was always positive, i.e., actual calculated price was greater than the new price using the Modified Duration relationship. In addition, the percentage changes in price are not symmetric. The percentage decrease in price for a given increase in yield is always less than the percent increase for the same decrease in yield. This property is refered to as convexity. Note that the two prices are quite close for small changes in the yield to maturity but the difference grows as the change in yield to maturity becomes bigger.

<u>Convexity.</u>

From Figure 2 it is clear that the Modified Duration relationship does not fully capture the true relationship between bond prices and yield to maturity. In order to more fully capture this, practitioners use Convexity. The definition of Convexity is

$$Convexity = CV = \frac{1}{P} \cdot \frac{\Delta^2 P}{(\Delta y)^2}$$

Once again those with a math background will recognize the last term on the right as the second derivative of price with respect to yield to maturity. The actual definition of Convexity that we can use is

$$Convexity = CV = \frac{1}{P}\cdot\frac{\Delta^2 P}{(\Delta y)^2} = \frac{1}{(1+y)^2}\cdot\sum_{t=1}^{T} t\cdot(t+1)\cdot\frac{C_t / (1+y)^t}{P}$$

Exhibit 4 shows the calculation of the semi-annual convexity for the six-year 6.1% coupon bond.

Exhibit 4

t	Cash Flow	Pres. Value Factor	Pres. Value of Cash Flow	Weight	Convexity $t \times (t+1) \times weight \times \frac{1}{(1+y)^2}$
1	$30.50	0.9524	$29.05	3.51%	0.0637
2	30.50	0.9070	27.66	3.34%	0.1820
3	30.50	0.8638	26.35	3.19%	0.3467
4	30.50	0.8227	25.09	3.03%	0.5503
5	30.50	0.7835	23.90	2.89%	0.7861
6	30.50	0.7462	22.76	2.75%	1.0482
7	30.50	0.7107	21.68	2.62%	1.3310
8	30.50	0.6768	20.64	2.50%	1.6298
9	30.50	0.6446	19.66	2.38%	1.9403
10	30.50	0.6139	18.72	2.26%	2.2585
11	30.50	0.5847	17.83	2.16%	2.5812
12	1,030.50	0.5568	573.82	69.37%	98.1588
	Bond Value		$827.17	Semi-annual Convexity	110.88

We can annualize the semi-annual convexity of 110.88 by dividing[5] it by 2^2 or 4. Here it would be 27.72. Convexity is useful to practitioners in a number of ways. First it can be used in conjunction with duration to get a more accurate estimate of the percentage price change resulting from a change in the yield. The formula[6] is

$$\%\ \Delta\ \text{Price} = -\ \text{Modified Duration} \times \Delta y + \frac{1}{2} \times \text{Convexity} \times (\Delta y)^2$$

[5] Convexity is annualized by dividing the calculated Convexity by the number of payments per year squared.

[6] For those with a math bent, this formula is based on using a Taylor series expansion to approximate the value of the percentage change in price.

Adding the convexity adjustment corrects for the fact that Modified Duration understates the true bond price. For example, in Exhibit 3, at a yield of 12% the percentage price change using only Modified Duration was -9.54%, while the actual was -9.01%. If we use the Convexity value we just calculated, the predicted percentage price change would be

$$\% \,\Delta \text{ Price} = -\,4.77 \times (.02) + \frac{1}{2} \times 27.72 \times (.02)^2 = -.0954 + .00554 = -.0899.$$

This is -8.99%, which is much closer to the actual percentage price change of -9.01%.

The pricing aspect of Convexity is much less important now since most people have access to calculators and computers that can do the pricing. The more important use of convexity is that it provides insight into how a bond will react to yield changes. Again from Exhibit 3 and Figure 2 we see that the price reaction to changes in yield is not symmetric. For a given change in yield, bond prices drop less for a given increase in yield and increase more for the same decreases in yield. The downside is less and the upside is more. This is clearly a desirable property. The higher the Convexity of a bond the more this is true. Thus, bonds with high convexity are more desirable.

Summary

Each of the price sensitivity measures discussed in this note is part of the everyday language and thinking of fixed income investors. They are relative risk measures that help investment professionals think about the risks they face.